3.3 Maximum and Minimum Values of a Function

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems:

- A farmer wants to choose the mix of crops that is likely to produce the largest profit.
- A doctor wishes to select the smallest dosage of a drug that will cure certain disease.
- The manufacturer would like to minimize the cost of distributing its products.

These problems can be reduced to finding the maximum or minimum values of a function. Let’s first explain exactly what we mean by the maximum and minimum values.

**Definition 3.1** A function $f$ has an **absolute maximum** (or **global maximum**) at $c$ if $f(c) \geq f(x)$ for all $x$ in $D$, where $D$ is the domain of $f$. The number $f(c)$ is called the **maximum value** of $f$ on $D$. Similarly, $f$ has an **absolute minimum** at $c$ if $f(c) \leq f(x)$ for all $x$ in $D$ and the number $f(c)$ is called the **minimum value** of $f$ on $D$. The maximum and minimum values of $f$ are called the **extreme values** of $f$.

Figure 3.1 shows the graph of a function $f$ with absolute maximum at $b$ and absolute minimum at $e$. Note that $(b, f(b))$ is the highest point on the graph and $(e, f(e))$ is the lowest point.

In general, there is no guarantee that a function will actually have an absolute maximum or minimum on the given interval.
The following theorem gives conditions under which a function is guaranteed to possess extreme values

**Theorem 3.1 (Extreme Value Theorem)**

*If* $f$ *is continuous on a closed interval* $[a, b]$, *then* $f$ *attains an absolute maximum value* $f(c)$ *and an minimum value* $f(d)$ *at some numbers* $c$ *and* $d$ *in* $[a, b]$.

The Extreme Value Theorem is illustrated in the following Figure. Note that an extreme value can be taken on more than once.

The following Figure show that a function need not be possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

**Definition 3.2** A function $f$ has a local maximum (or relative maximum) at $c$ if $f(c) \geq f(x)$ when $x$ is near $c$ [This means that $f(c) \geq f(x)$ for all $x$ in some open interval containing $c$.] Similarly, $f$ has a local minimum at $c$ if $f(c) \leq f(x)$ when $x$ is near $c$. If $f$ has either a local maximum or a local minimum at $c$, then $f$ is said to have a local extreme values at $c$. 
Local max \[ f'(a) \text{ does not exist} \]

Local min \[ f'(b) \text{ does not exist} \]

Local max \[ f'(c) = 0 \]

Local min \[ f'(d) = 0 \]

Figure 3.2: Local extreme values

Figure 3.2 illustrates that a local extreme value can occur at a point in the domain of function at which either the graph of the function has a horizontal tangent line or function is not differentiable.

**Definition 3.3** A critical number of a function \( f \) is a number \( c \) in the domain of \( f \) such that \( f'(c) = 0 \) or \( f'(c) \text{ does not exist} \).

**Theorem 3.2 (Fermat’s Theorem)** If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a critical number of \( f \).

**Example 3.17** Find all critical numbers and the local extreme values of \( f(x) = 2x^3 + 3x^2 - 12x - 5 \).

**Solution**

**Example 3.18** Find all critical numbers and the local extreme values of \( f(x) = (2x + 3)^{2/3} \).

**Solution**

Fermat’s Theorem does suggest that we should at least start looking for extreme values of \( f \) at a critical number of \( f \).

**Example 3.19** Find the critical numbers of

\[ f(x) = \frac{x^2 + 3}{x + 1}. \]

**Solution**

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number] or it occurs at an endpoint of the interval. Thus the following three-step procedure always works.
The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function $f$ on a closed interval $[a, b]$:

1. Find the critical points of $f$ in $(a, b)$
2. Evaluate $f$ at all the critical numbers and at the endpoints $a$ and $b$.
3. The largest of the values in Step 2 is the absolute maximum value of $f$ on $[a, b]$ and the smallest value is the absolute minimum.

Example 3.20 Find the absolute maximum and absolute minimum values of the function

$$f(x) = x^3 - 3x + 1$$

on the interval $[0, 3]$.

Solution .........

Example 3.21 Find the absolute maximum and absolute minimum values of the function

$$f(x) = \frac{\ln x}{x} \quad 1 \leq x \leq 3.$$

Solution .........

3.4 Increasing and Decreasing Functions

Definition 3.4 Let $f$ be defined on an interval $I$ (open, closed, or neither), and let $x_1$ and $x_2$ denote points in $I$.

(a) $f$ is increasing on $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
(b) $f$ is decreasing on $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
(c) $f$ is constant on $I$ if $f(x_1) = f(x_2)$ for all points $x_1$ and $x_2$.

Theorem 3.3 (Increasing/Decreasing Test) Let $f$ be continuous on an interval $I$ and differentiable at every interior point of $I$.

(i) If $f'(x) > 0$ for all $x$ interior to $I$, then $f$ is increasing on $I$.
(ii) If $f'(x) < 0$ for all $x$ interior to $I$, then $f$ is decreasing on $I$.
(iii) If $f'(x) = 0$ for all $x$ in $I$, then $f$ is constant on $I$.

Example 3.22 Find the interval on which $f(x) = 3x^4 + 4x^3 - 12x^2 - 5$ is increasing and the interval on which it is decreasing.

Solution .........
**Theorem 3.4 (First Derivative Test)**

Let $f$ be continuous on an open interval $(a, b)$ that contains a critical number $c$.

1. If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then $f(c)$ is a local maximum value of $f$.

2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then $f(c)$ is a local minimum value of $f$.

3. If $f'(x)$ has the same sign on both sides of $c$, then $f(c)$ is not a local extreme value of $f$.

It is easy to remember the First Derivative Test by visualizing diagram such as those in the following Figure.

**Example 3.23** Find the local minimum and maximum values of the function $f(x) = xe^x$.

**Solution** ......

**Example 3.24** Find the local minimum and maximum values of $f(x) = x(x - 1)^3$.

**Solution** ......

### 3.5 Concavity

**Definition 3.5** Let $f$ be differentiable on an open interval $I$. We say that $f$ (as well as its graph) is **concave up** on $I$ if $f'$ is increasing on $I$ and we say that $f$ is **concave down** on $I$ if $f'$ is decreasing on $I$. 
Theorem 3.5 (Concavity Test)

Let \( f \) be twice differentiable on an open interval \( I \).

1. If \( f'' > 0 \) for all \( x \) in \( I \), then \( f \) is concave up on \( I \).

2. If \( f'' < 0 \) for all \( x \) in \( I \), then \( f \) is concave down on \( I \).

Definition 3.6 Let \( f \) be continuous at \( c \). We call \((c, f(c))\) an inflection point of the graph \( f \) if \( f \) is concave up on one side of \( c \) and concave down on the other side.

Example 3.25 Let \( f(x) = x^4 - 6x^2 + 3 \). Find the intervals of concavity and the inflection points.

Solution ...........

Theorem 3.6 (The Second Derivative Test) Let \( f' \) and \( f'' \) exist at every point in an open interval \((a, b)\) containing \( c \), and suppose \( f'(c) = 0 \).

1. If \( f''(c) < 0 \), then \( f(c) \) is a local maximum value of \( f \).

2. If \( f''(c) > 0 \), then \( f(c) \) is a local minimum value of \( f \).

Example 3.26 For \( f(x) = 2x^3 + 6x^2 - 18x + 5 \), use the Second Derivative Test to identify local extrema.

Solution ...........

Note. The Second Derivative Test is inconclusive when \( f''(c) = 0 \). In other words, at such a point there might be a maximum, there might be a minimum, or there be neither. This test also fails when \( f''(c) \) does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 3.27 Let \( f(x) = x^4 - 4x^3 + 10 \).

- Find the interval of increase and decrease.
- Find the local maximum and minimum values.
- Find the intervals of concavity and the inflection points.
- Use the above information to sketch the graph.

Solution ...........

Example 3.28 Let \( f(x) = x^{2/3}(6 - x)^{1/3} \).

- Find the interval of increase and decrease.
- Find the local maximum and minimum values.
- Find the intervals of concavity and the inflection points.
- Use the above information to sketch the graph.

Solution .............
3.6 Applied Maximum and Minimum Problems

In this section we will show how the methods discussed in the preceding section can be used to solve various applied optimization problems.

A Procedure for Solving Applied Maximum and Minimum Problems

1. Read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

2. Draw an appropriate figure and label the quantities relevant to the problem.

3. Find a formula for the quantity to be maximized or minimized.

4. Using the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

5. Find the interval of possible values for this variable from the physical restrictions in the problem.

6. If applicable, use the techniques of the preceding section to obtain the maximum or minimum.

Example 3.29 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution ...........

Example 3.30 Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.

Solution ...........

Example 3.31 A rectangular beam is to be cut from a log with circular cross section. If the strength of the beam is proportional to the product of its width and the square of its depth, find the dimensions of the cross section that give the strongest beam.

Solution ...........

3.7 Rolle’s Theorem; Mean Value Theorem

In this section we will discuss a result called the Mean Value Theorem. The theorem has so many important consequences that it is regarded as one of the major principle is calculus

Theorem 3.7 (Rolle's Theorem) Let $f$ be a function that satisfies the following three hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$.

2. $f$ is differentiable on the open interval $(a, b)$.
3. \( f(a) = f(b) \)

Then there is a number \( c \in (a, b) \) such that \( f'(c) = 0 \).

The following Figure shows the graphs of three such functions. In each case it appears that there is at least one point \( (c, f(c)) \) on the graph where the tangent is horizontal and therefore \( f'(c) = 0 \).

![Graphs of three functions](image)

**Example 3.32** Verify that the function

\[
f(x) = x^3 - 3x^2 + 2x + 2, \quad [0, 1]
\]

satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers \( c \) that satisfy the conclusion of Rolle's Theorem.

**Solution** .........

**Theorem 3.8 (Mean Value Theorem)** Let \( f \) be a function that satisfies the following hypotheses:

1. \( f \) is continuous on the closed interval \([a, b]\).
2. \( f \) is differentiable on the open interval \((a, b)\).

Then there is a number \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

or, equivalently,

\[
f(b) - f(a) = f'(c)(b - a)
\]

**Example 3.33** Let

\[
f(x) = x^3 - x^2 - x + 1, \quad [-1, 2]
\]

Find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

**Solution** .........
Chapter 4
Integration

4.1 Antiderivatives; The Indefinite Integral

Antiderivatives

Definition 4.1 A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

For instance, let $f(x) = x^2$. It isn’t difficult to discover an antiderivative $F(x) = \frac{1}{3}x^3$ because $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore, both $F$ and $G$ are antiderivatives of $f$. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where $C$ is a constant, is an antiderivative of $f$.

Question! Are there any others?

Answer. No.

Thus, if $F$ and $G$ are any two antiderivatives of $f$, then

$$F'(x) = f(x) = G'(x)$$

so $G(x) - F(x) = C$, where $C$ is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

Theorem 4.1 If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C$$

where $C$ is an arbitrary constant.

The Indefinite Integral

The process of finding antiderivatives is called antidifferentiation or integration. Thus, if

$$\frac{d}{dx}[F(x)] = f(x)$$

(4.1)
then integrating (or antidifferentiating) the function \( f(x) \) produces an antiderivative of the form \( F(x) + C \). To emphasize this process, Equation (4.1) is recast using integral notation.

\[
\int f(x) \, dx = F(x) + C
\]  

(4.2)

where \( C \) is an arbitrary constant. For example,

\[
\int x^2 \, dx = \frac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx} \left[ \frac{1}{3}x^3 \right] = x^2
\]

Note that if we differentiate an antiderivative of \( f(x) \), we obtain \( f(x) \) back again. Thus,

\[
\frac{d}{dx} \left[ \int f(x) \, dx \right] = f(x)
\]  

(4.3)

The expression \( \int f(x) \, dx \) is called an indefinite integral. The “elongated s” that appears on the left side of (4.2) is called an integral sign, the function \( f(x) \) is called the integrand, and the constant \( C \) is called the constant of integration.

The differential symbol, \( dx \), in the differentiation and antidifferentiation operations

\[
\frac{d}{dx} [ \quad ] \quad \text{and} \quad \int [ \quad ] \, dx
\]

serves to identify the independent variable. If an independent variable other than \( x \) is used, say \( t \), then the notation must be adjusted appropriately. Thus,

\[
\frac{d}{dt} [F(t)] = f(t) \quad \text{and} \quad \int [f(t)] \, dx = F(t) + C
\]

are equivalent statements. Here are some examples of derivative formulas and their equivalent integration formulas:

<table>
<thead>
<tr>
<th>Derivative Formula</th>
<th>Equivalent Integration Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} [x^3] = 3x^2 )</td>
<td>( \int 3x^2 , dx = x^3 + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx} [\sqrt{x}] = \frac{1}{2\sqrt{x}} )</td>
<td>( \int \frac{1}{2\sqrt{x}} , dx = \sqrt{x} + C )</td>
</tr>
<tr>
<td>( \frac{d}{dt} [\tan t] = \sec^2 t )</td>
<td>( \int \sec^2 t , dt = \tan t + C )</td>
</tr>
</tbody>
</table>

**Integration Formulas**

Some of the most important integration formulas are given in the following Table.
\[
\begin{align*}
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \\
\int e^x \, dx &= e^x + C \\
\int \ln a \, dx &= \frac{a^x}{\ln a} + C, \quad a > 0, \quad a \neq 1 \\
\int \sin x \, dx &= -\cos x + C \\
\int \sec^2 x \, dx &= \tan x + C \\
\int \csc^2 x \, dx &= -\cot x + C \\
\int \sec x \tan x \, dx &= \sec x + C \\
\int \csc x \cot x \, dx &= -\csc x + C \\
\int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1} x + C \\
\int \frac{1}{1+x^2} \, dx &= \tan^{-1} x + C \\
\int \frac{1}{|x|\sqrt{x^2-1}} \, dx &= \sec^{-1} x + C
\end{align*}
\]

**Properties of the Indefinite Integral**

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivative.

**Theorem 4.2** Let \( f \) and \( g \) have antiderivatives (indefinite integrals) and let \( c \) be a constant. Then

\begin{enumerate}
  \item \( \int c f(x) \, dx = c \int f(x) \, dx \)
  \item \( \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \)
  \item \( \int [f(x) - g(x)] \, dx = \int f(x) \, dx - \int g(x) \, dx \)
\end{enumerate}

**Example 4.1** Evaluate \( \int (3e^x + 5x^{2/3}) \, dx \).

**Solution** 

**Example 4.2** Evaluate \( \int (3\cos x + 2\sec^2 x) \, dx \).

**Solution** 

**Example 4.3** Evaluate

\begin{enumerate}
  \item \( \int \frac{\cos x}{\sin^2 x} \, dx \)
  \item \( \int \frac{2t^4 - t^2 \sqrt{t} - 1}{t^4} \, dt \)
\end{enumerate}

**Solution** 

4.2 Integration by Substitution

In this section we shall discuss a technique, called substitution, which can often be used to transform complicated integration problems into simpler ones.

\textbf{u-substitution}

The method of substitution hinges on the following formula in which \(u\) stands for a differentiable function of \(x\).

\[
\int \left[ f(u) \frac{du}{dx} \right] dx = \int f(u) du \tag{4.4}
\]

To justify this formula, let \(F\) be an antiderivative of \(f\), so that

\[
\frac{d}{du}[F(u)] = f(u)
\]
or, equivalently,

\[
\int f(u) du = F(u) + C \tag{4.5}
\]

If \(u\) is a differentiable function of \(x\), the chain rule implied that

\[
\frac{d}{dx}[F(u)] = \frac{d}{du}[F(u)] \cdot \frac{du}{dx} = f(u) \frac{du}{dx}
\]
or, equivalently,

\[
\int \left[ f(u) \frac{du}{dx} \right] dx = F(u) + C \tag{4.6}
\]

Formula (4.4) follows from (4.5) and (4.6).

The following example illustrates how Formula (4.4) is used.

\textbf{Example 4.4} Evaluate \( \int (x^4 - 1)^{99}(4x^3) \, dx \).

\textbf{Solution} ............

In general, suppose that we are interested in evaluating

\[
\int h(x) \, dx
\]

It follows from (4.4) that if we can express this integral in the form

\[
\int h(x) \, dx = \int f(g(x))g'(x) \, dx
\]

then the substitution \(u = g(x)\) and \(du/dx = g'(x)\) will yield

\[
\int h(x) \, dx = \int \left[ f(u) \frac{du}{dx} \right] dx = \int f(u) \, du
\]

With a “good” choice of \(u = g(x)\), the integral on the right will be easier to evaluate than the original.

In practice, this substitution process is carried out as follows:
Guideline for \( u \)-Substitution

**Step 1.** Make a choice for \( u \), say \( u = g(x) \).

**Step 2.** Compute \( du/dx \)

**Step 3.** Make the substitution \( u = g(x) \), \( du = g'(x) \, dx \)

At this stage, the entire integral must be in terms of \( u \); no \( x \)'s should remain. If this is not the case, try a different choice of \( u \).

**Step 4.** Evaluate the resulting integral.

**Step 5.** Replace \( u \) by \( g(x) \), so the final answer is in terms of \( x \).

**Example 4.5** Evaluate \( \int x^2 \cos(x^3 - 2) \, dx \).

**Solution** ...........

**Example 4.6** Evaluate \( \int \sqrt{2 \sin x + 1} \cos x \, dx \).

**Solution** ...........

**Example 4.7** Evaluate \( \int \frac{x}{\sqrt{1 - 4x^2}} \, dx \).

**Solution** ...........

**Example 4.8** Evaluate \( \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \).

**Solution** ...........

**Example 4.9** Evaluate \( \int \sqrt{1 + x^2} x^5 \, dx \).

**Solution** ...........

### 4.3 The Definite Integral

**Definition of Area**

The first goal in this section is to give a mathematical definition of *area*. We begin by attempting to find the area of the region \( S \) that lies under the curve \( y = f(x) \) from \( a \) to \( b \). In general, we start by subdividing \( S \) into \( n \) strips \( S_1, S_2, \ldots, S_n \) of equal width as in Figure.
The width of the interval \([a, b]\) is \(b - a\), so the width of each of the \(n\) strips is

\[
\Delta x = \frac{b - a}{n}
\]

These strips divide the interval \([a, b]\) into \(n\) subintervals

\([x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]\)

where \(x_0 = a\) and \(x_n = b\). The right-hand endpoints of the subintervals are

\[
x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots
\]

Let’s approximate the \(i\)th strip \(S_i\) by a rectangle with width \(\Delta x\) and height \(f(x_i)\), which is the value of \(f\) at the right-hand endpoint. Then the area of the \(i\)th rectangle is \(f(x_i)\Delta x\). What we thing of intuitively as the area of \(S\) is approximated by the sum of the areas of these rectangles, which is

\[
R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x
\]

\[
= \sum_{i=1}^{n} f(x_i)\Delta x
\]

Notice that this approximation appears to become better and better as the number of strips increases, that is, as \(n \to \infty\). Therefore, we define the area \(A\) of the region \(S\) in the following way.

**Definition 4.2** The **area** \(A\) of the region \(S\) that lies under the graph of the continuous function \(f\) is the limit of the sum of the areas of approximating rectangles:

\[
A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x
\]
We now have that a limit of the form
\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to \infty} \left[ f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right] \]
arises when we compute an area. However this type of limit occurs in a wide variety of situations even when \( f \) is not necessarily a positive function. In the next Chapter we will see that limit of this form also arise in finding volumes of solids. We therefore give type of limit a special name and notation.

**Definition 4.3** If \( f \) is a continuous function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a) / n \). We let \( x_0 = a \), \( x_1, x_2, \ldots, x_n = b \) be the endpoints of these subintervals and we choose simple points \( x_1^*, x_2^*, \ldots, x_n^* \) in these subintervals, so \( x_i^* \) lies in the \( i \)th subinterval \([x_{i-1}, x_i]\). Then the definite integral of \( f \) from \( a \) to \( b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x
\]

(4.7)

**Note:**

1. The symbol \( \int \) was introduced by Leibniz and is called an integral sign. In the notation \( \int_{a}^{b} f(x) \, dx \), \( f(x) \) is called integrand and \( a \) and \( b \) are called the limit of integration; \( a \) is the lower limit and \( b \) is the upper limit. The symbol \( dx \) has no official meaning by itself; \( \int_{a}^{b} f(x) \, dx \) is all one symbol. The procedure of calculating an integral is called integration.

2. The definite integral \( \int_{a}^{b} f(x) \, dx \) is a number; it does not depend on \( x \). In fact, we could use any letter in place of \( x \) without changing the value of the integral:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt = \int_{a}^{b} f(r) \, dr
\]

3. Because we have assumed that \( f \) is continuous, it can be proved that the limit in Definition 4.3 always exists and gives the same value no matter how we choose the sample points \( x_i^* \).

4. The sum

\[
\sum_{i=1}^{n} f(x_i^*)\Delta x
\]

that occurs in Definition 4.3 is called Riemann sum after the German mathematician Bernhard Riemann (1826 - 1866). We know that if \( f \) happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (See Figure 4.1 (a)). By comparing Definition 4.3 with the definition of area, we see that the definite integral \( \int_{a}^{b} f(x) \, dx \) can be interpreted as the area under the curve \( y = f(x) \) from \( a \) to \( b \) (Figure 4.1 (b)).
If \( f \) take on both positive and negative values, then the Riemann sum is the sum of areas of the rectangles that lie above the \( x \)-axis and the negative of the areas of the rectangles that lie below the \( x \)-axis. When we take the limit of such Riemann sums, we get the situation illustrated in the following Figure.

A definite integral can be interpreted as a \textit{net area}, that is, a difference of areas:

\[
\int_{a}^{b} f(x) \, dx = A_1 - A_2
\]

where \( A_1 \) is the area of the region above the \( x \)-axis and below the graph of \( f \) and \( A_2 \) is the area of the region below the \( x \)-axis and above the graph of \( f \).

5. Although we have defined \( \int_{a}^{b} f(x) \, dx \) by dividing \([a, b]\) into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

When we defined the definite integral, we implicitly assumed that \( a < b \). But the definition as a limit of Riemann sums makes sense even if \( a > b \). Notice that if we reverse \( a \) and \( b \), than \( \Delta x \) changes from \((b - a)/n\) to \((a - b)/n\). Therefore

\[
\int_{b}^{a} f(x) \, dx = - \int_{a}^{b} f(x) \, dx
\]

If \( a = b \), then \( \Delta x = 0 \) and so

\[
\int_{a}^{a} f(x) \, dx = 0
\]

We now develop some basic properties of integrals. We assume that \( f \) and \( g \) are continuous on an interval \([a, b]\).
Theorem 4.3 (Properties of the Integral)

1. \[ \int_a^b c \, dx = c(b - a), \] where \( c \) is any constant

2. \[ \int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx, \] where \( c \) is any constant

3. \[ \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

4. \[ \int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \]

Theorem 4.4 If \( f \) and \( g \) are continuous on an interval \([a, b]\) and \( c \) is any constant, then

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx. \]

For the case where \( f(x) \geq 0 \) and \( a < c < b \), this theorem can be seen from the geometric interpretation in the following Figure.

The area under \( y = f(x) \) from \( a \) to \( c \) plus the area from \( c \) to \( b \) is equal to the total area from \( a \) to \( b \).

Theorem 4.5 Let \( f \) and \( g \) be continuous on an interval \([a, b]\) and \( g(x) \leq f(x) \) for all \( x \) in \([a, b]\). Then

\[ \int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx \]

Example 4.10 If it is known that \( \int_0^8 f(x) \, dx = 3, \int_0^5 f(x) \, dx = -4, \int_5^6 g(x) \, dx = 1, \) and \( \int_6^8 g(x) \, dx = -2, \) find \( \int_5^8 (2f(x) + g(x)) \, dx. \)

Solution . . . . . .
4.4 The Fundamental Theorem of Calculus

In the previous section we defined the concept of the definite integral but did not give any general methods for evaluating them. In this section we shall give a method for using antiderivatives to evaluate definite integrals.

**Theorem 4.6 (The First Fundamental Theorem of Calculus).** If \( f \) is continuous on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a) = F(x) \bigg|_{a}^{b}
\]

where \( F \) is any antiderivative of \( f \), that is, a function such that \( F' = f \).

**Example 4.11** Evaluate \( \int_{0}^{1} (3 + x\sqrt{x}) \, dx \).

**Solution** .......

**Example 4.12** Evaluate \( \int_{1}^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} \, d\theta \).

**Solution** .......

**Theorem 4.7 (The Second Fundamental Theorem of Calculus).** If \( f \) is continuous on \([a, b]\), then the function \( F \) defined by

\[
F(x) = \int_{a}^{x} f(t) \, dt, \quad a \leq x \leq b
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( F'(x) = f(x) \).

Using Leibniz notation for derivative, the result in Theorem 4.7 can be expressed by the formula

\[
\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x)
\]

**Example 4.13** Find the derivative of the function \( g(x) = \int_{0}^{x} \sqrt{1 + t^2} \, dt \).

**Solution** .......

**Example 4.14** Find the derivative of the function \( g(x) = \int_{1}^{x^4} \sec t \, dt \).

**Solution** .......

**Example 4.15** Let \( F(x) = \int_{x^2}^{x^3} \left(e^t + 1\right) \, dt \). Find \( F'(x) \).

**Solution** .........
4.5 Evaluating Definite Integrals by Substitution

There is only one slight difference in using substitution for evaluating a definite integral: If you change variables, you must also change the limits of integration to correspond to the new variable. That is, when you introduce the new variable \( u = g(x) \), you must also change the limits of integration from \( x = a \) and \( x = b \) to the corresponding limits for \( u : u = g(a) \) and \( u = g(b) \). We have

\[
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

**Example 4.16** Evaluate \( \int_{1}^{2} x^3 \sqrt{x^4 + 5} \, dx \).

**Solution** . . . . . .

**Example 4.17** Evaluate \( \int_{1}^{e} \frac{\ln x}{x} \, dx \).

**Solution** . . . . . .

**Example 4.18** Evaluate \( \int_{1}^{3} \frac{\cos(\pi/x)}{x^2} \, dx \).

**Solution** . . . . . .
Chapter 5

Applications of Definite Integral

5.1 Area Between Two Curves

In this section we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region that lies between two curves \( y = f(x) \) and \( y = g(x) \) and between the vertical lines \( x = a \) and \( x = b \), where \( f \) and \( g \) are continuous functions and \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\).

The area \( A \) of this region is

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

(5.1)

Example 5.1 Find the area of the region enclosed by the line \( y = 3 - x \) and the parabola \( y = x^2 - 9 \).

Solution ...........

Example 5.2 Find the area of the region bounded by the parabolas \( y = x^2 \) and \( y = 2x - x^2 \).

Solution ...........

Example 5.3 Find the area bounded by the graphs of \( y = x^2 \) and \( y = 2 - x^2 \) for \( 0 \leq x \leq 2 \).

Solution ............
Some regions are best treated by regarding $x$ as a function of $y$. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where $f$ and $g$ are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

$$A = \int_{c}^{d} [f(y) - g(y)] \, dy$$  \hfill (5.2)

**Example 5.4** Find the area of the region bounded by the graphs of $y = x^2$, $y = 2 - x$, and $y = 0$.

**Solution** ............

**Example 5.5** Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$.

**Solution** ............

**Example 5.6** Find the area of the region bounded by the curves $x = y^2$, $y = x + 5$, $y = 2$, and $y = -1$.

**Solution** ............

### 5.2 Volumes by Slicing: Disks and Washers

In this section we will use definite integral to find volumes of solid of revolution.

**Method of Disks**

Suppose that $f(x) \geq 0$ and $f$ is continuous on $[a, b]$. Take the region bounded by the curve $y = f(x)$ and the $x$-axis, for $a \leq x \leq b$ and revolve it about the $x$-axis, generating a solid.
We can find the volume of this solid by slicing it perpendicular to the $x$-axis and recognizing that each cross section is a circular disk of radius $r = f(x)$. We then have that the volume of the solid is

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

cross-sectional area $= \pi r^2$

**Example 5.7** Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$ from $0$ to $4$ about the $x$-axis.

**Solution** .........

In a similar way, suppose that $g(y) \geq 0$ and $g$ is continuous on the interval $[c, d]$. Then, revolving the region bounded by the curve $x = g(y)$ and the $y$-axis, for $c \leq y \leq d$, about the $y$-axis generates a solid.

Once again, notice from Figure that the cross sections of the resulting solid of revolution are circular disks of radius $r = g(y)$. The volume of the solid is then given by

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

cross-sectional area $= \pi r^2$

**Example 5.8** Find the volume of the solid obtained by rotating the region bounded by $y = 2 - \frac{x^2}{2}$ from $x = 0$ to $x = 2$ about the $y$-axis.

**Solution** .........
Method of Washers

There are two complications that can be found in the types of volume calculations we have been studying. The first of these is that you may need to compute the volume of a solid that have a cavity or “hole” in it. The second of these occurs when a region is revolved about a line other that the \( x \)-axis or the \( y \)-axis.

Suppose that \( f \) and \( g \) are nonnegative continuous function such that

\[
g(x) \leq f(x) \quad \text{for} \quad a \leq x \leq b
\]

and let \( R \) be the region enclosed between the graphs of these functions and the lines \( x = a \) and \( x = b \).

When this region is revolved about the \( x \)-axis, it generates a solid having annular or washer-shaped cross sections. Since the cross section at \( x \) has inner radius \( g(x) \) and outer radius \( f(x) \), its volume of the solid is

\[
V = \int_{a}^{b} \pi \left( f(x)^2 - g(x)^2 \right) dx
\]

Suppose that \( u \) and \( v \) are nonnegative continuous function such that

\[
v(y) \leq u(y) \quad \text{for} \quad c \leq y \leq d.
\]

If \( R \) the region enclosed between the graphs of \( x = u(y) \) and \( x = v(y) \) and the lines \( y = c \) and \( y = d \).

When this region is revolved about the \( y \)-axis, it also generates a solid having annular or washer-shaped cross sections. Since the cross section at \( y \) has inner radius \( v(y) \) and outer radius \( u(y) \), its volume of the solid is

\[
V = \int_{c}^{d} \pi \left( u(y)^2 - v(y)^2 \right) dy
\]
Example 5.9 The region \( R \) enclosed by the curves \( y = 4 - x^2 \) and \( y = 0 \). Find the volume of the solid obtained by rotating the region \( R \) (a) about the \( y \)-axis (b) about the line \( y = -3 \) (c) about the line \( y = 7 \), and (d) about the line \( x = 3 \).

Solution .........

Example 5.10 Find the volume of the solid obtained by rotating the region bounded by \( y = 1 + \frac{x^2}{2} \) and the line \( y = 2 \) about the line \( y = -2 \).

Solution .........

5.3 Volumes by Cylindrical Shells

A cylindrical shells is a solid enclosed by two concentric right-circular cylinders.

\[
\text{The volume } V \text{ of a cylindrical shell having inner radius } r_1, \text{ outer radius } r_2, \text{ and height } h \text{ can be written as}
\]

\[
V = \left[ \text{area of cross section} \right] \cdot \left[ \text{height} \right] = (\pi r_2^2 - \pi r_1^2)h = \pi (r_2 + r_1)(r_2 - r_1)h = 2\pi \cdot \left( \frac{r_2 + r_1}{2} \right) \cdot h \cdot (r_2 - r_1)
\]

But \( \frac{r_2 + r_1}{2} \) is the average radius of the shell and \( r_2 - r_1 \) is its thickness, so

\[
V = 2\pi \cdot \left[ \text{average radius} \right] \cdot \left[ \text{height} \right] \cdot \left[ \text{thickness} \right]
\]

This formula can be used to find the volume of a solid of revolution.

Let \( R \) be a plane region bounded above by a continuous curve \( y = f(x) \), bounded below by the \( x \)-axis, and bounded on the left and right, respectively, by the line \( x = a \) and \( x = b \).

The volume of the solid generated by revolving \( R \) about the \( y \)-axis is given by

\[
V = 2\pi \int_a^b \frac{x}{\text{radius}} f(x) \frac{dx}{\text{height}} \cdot \frac{dx}{\text{thickness}}
\]

Let \( R \) be a plane region bounded above by a continuous curve \( x = g(y) \), the \( y \)-axis, and the line \( y = c \) and \( y = d \).
The volume of the solid generated by revolving $R$ about the $x$-axis is given by

$$V = 2\pi \int_c^d y \cdot g(y) \, dy$$

**Example 5.11** Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ in the first quadrant about the $y$-axis.

**Solution**

**Example 5.12** The region bounded by the line $y = \left(\frac{r}{h}\right) x$, the $x$-axis, and $x = h$ is revolved about the $x$-axis, thereby generating cone (assume $r > 0$, $h > 0$). Find its volume by the shell method.

**Solution**

**5.4 Length of a Plane Curve**

**Arc Length Problem.** Suppose $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Find the arc length $L$ of the curve $y = f(x)$ over the interval $[a, b]$.

In order to solve this problem, we begin by partitioning the interval $[a, b]$ into $n$ equal pieces:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where

$$x_i - x_{i-1} = \Delta x = \frac{b - a}{n},$$
for each \( i = 1, 2, \ldots, n \).

Between each pair of adjacent points on the curve, \((x_{i-1}, f(x_{i-1}))\) and \((x_i, f(x_i))\) we approximate the arc length \( \ell_i \) by the straight-line distance between the two points.

From the usual distance formula, we have

\[
\ell_i \approx \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}.
\]

Since \( f \) is continuous on all of \([a, b]\) and differentiable on \((a, b)\), \( f \) is also continuous on the subinterval \([x_{i-1}, x_i]\) and is differentiable on \((x_{i-1}, x_i)\). Recall that by the Mean Value Theorem, we have

\[
f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),
\]

for some number \( c_i \in (x_{i-1}, x_i) \). This gives us the approximation

\[
\ell_i \approx \sqrt{(x_i - x_{i-1})^2 + [f'(c_i)(x_i - x_{i-1})]^2}
\]

\[
= \sqrt{1 + [f'(c_i)]^2 (x_i - x_{i-1})}
\]

Adding together the lengths of these \( n \) line segments, we get an approximation of the total arc length,

\[
L \approx \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\Delta x.
\]

Notice that as \( n \) gets larger, this approximation should approach the exact arc length, that is,

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\Delta x.
\]

You should recognize this as the limit of a Riemann sum for \( \sqrt{1 + [f'(x)]^2} \), so that the arc length is given by the definite integral:

\[
L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} dx,
\]

whenever the limit exists.
Example 5.13  *Find the arc length of the curve* \( y = x^{3/2} \) *from* \((1, 1)\) *to* \((4, 8)\).

**Solution** ........
Chapter 6

Techniques of Integration

6.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The Product Rule states that if $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)] \, dx = f(x)g(x)$$

or

$$\int f(x)g'(x) \, dx + \int g(x)f'(x) \, dx = f(x)g(x)$$

We can rearrange this equation as

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int g(x)f'(x) \, dx$$

This formula is called the formula for integration by parts. Let $u = f(x)$ and $v = g(x)$. Then the differential are $du = f'(x) \, dx$ and $dv = g'(x) \, dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\int u \, dv = uv - \int v \, du$$
Example 6.1 Find $\int x \cos x \, dx$.

Solution ............

Note. Our aim in using integration by parts is to obtain a simpler integral than the one we start with. Thus, in Example 6.1 we start with $\int x \cos x \, dx$ and expressed it in terms of the simpler integral $\int \sin x \, dx$. If we had chosen $u = \cos x$ and $dv = x \, dx$, then $du = -\sin x \, dx$ and $v = \frac{x^2}{2}$, so integration by parts gives

$$\int x \cos x \, dx = (\cos x) \frac{x^2}{2} + \frac{1}{2} \int x^2 \sin x \, dx$$

Although this is true, $\int x^2 \sin x \, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for $u$ and $dv$, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x) \, dx$ can be readily integrated to give $v$.

Example 6.2 Find $\int x \ln x \, dx$.

Solution ............

Example 6.3 Find $\int x^2 \sin 2x \, dx$.

Solution ............

Example 6.4 Find $\int e^{2x} \cos x \, dx$.

Solution ............

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Assuming $f'$ and $g'$ are continuous, and using the Fundamental Theorem, we obtain

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \bigg|_a^b - \int_a^b g(x)f'(x) \, dx$$

That is, if $u = f(x)$ and $v = g(x)$, then

$$\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du.$$

Example 6.5 Find $\int_0^{1/3} \tan^{-1} 3x \, dx$.

Solution ............
6.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions. Our first aim is to evaluate integrals of the form
\[ \int \sin^m x \cos^n x \, dx, \]
where \( m \) and \( n \) are positive integers.

**Case I: \( m \) or \( n \) Is an Odd Positive Integer**

If \( m \) is odd, first isolate one factor of \( \sin x \). (You’ll need this for \( du \)). Then, replace any factors of \( \sin^2 x \) with \( 1 - \cos^2 x \) and make the substitution \( u = \cos x \). Likewise, if \( n \) is odd, first isolate one factor of \( \cos x \). (You’ll need this for \( du \)). Then, replace any factors of \( \cos^2 x \) with \( 1 - \sin^2 x \) and make the substitution \( u = \sin x \).

**Example 6.6** Evaluate \( \int \sin^3 x \, dx \).

**Solution** ………

**Example 6.7** Find \( \int \cos^4 x \sin^3 x \, dx \).

**Solution** ………

**Example 6.8** Find \( \int \sqrt{\sin x} \cos^5 x \, dx \).

**Solution** ………

**Case II: \( m \) and \( n \) Are Both Even Positive Integers**

In this case, we can use the half-angle formulas for sine and cosine to reduce the power of in the integrand.

**Half-angle formulas:**
\[ \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \quad \text{and} \quad \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \]

**Example 6.9** Find \( \int \cos^4 x \, dx \).

**Solution** ………

**Example 6.10** Find \( \int \sin^2 x \cos^4 x \, dx \).

**Solution** ………

Our next aim is to devise a strategy for evaluating integrals of the form
\[ \int \tan^m x \sec^n x \, dx, \]
where \( m \) and \( n \) are integers.
Case I: \( m \) Is an Odd Positive Integer

First, isolate one factor of \( \sec x \tan x \). (You’ll need this for \( du \)). Then, replace any factors of \( \tan^2 x \) with \( \sec^2 x - 1 \) and make the substitution \( u = \sec x \).

**Example 6.11** Find \( \int \tan^3 x \sec^{3/2} x \, dx \).

**Solution**

Case II: \( n \) Is an Even Positive Integer

First, isolate one factor of \( \sec^2 x \). (You’ll need this for \( du \)). Then, replace any remaining factors of \( \sec^2 x \) with \( 1 + \tan^2 x \) and make the substitution \( u = \tan x \).

**Example 6.12** Find \( \int \tan^4 x \sec^4 x \, dx \).

**Solution**

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by part, and occasionally a little ingenuity. We will sometimes need to be able to integrate \( \tan x \) by using the formula

\[
\int \tan x \, dx = \ln |\sec x| + C
\]

We will also need the indefinite integral of secant:

\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

**Example 6.13** Find \( \int \tan^3 x \, dx \).

**Solution**

**Example 6.14** Find \( \int \sec^3 x \, dx \).

**Solution**

Integrals of the form \( \int \cot^m x \csc^n x \, dx \) can be found by similar methods because of the identity

\[
1 + \cot^2 x = \csc^2 x.
\]

Finally, we can make use of another set of trigonometric identities: To evaluate the integrals

(a) \( \int \sin mx \cos nx \, dx \),

(b) \( \int \sin mx \sin nx \, dx \), or
(c) \( \int \cos mx \cos nx \, dx \),

we use the corresponding identity:

\[
\sin A \cos B = \frac{1}{2} \left[ \sin(A - B) + \sin(A + B) \right]
\]

\[
\sin A \sin B = \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right]
\]

\[
\cos A \cos B = \frac{1}{2} \left[ \cos(A - B) + \cos(A + B) \right]
\]

**Example 6.15** Find \( \int \sin 5x \cos 6x \, dx \).

**Solution**

**Example 6.16** Find \( \int \cos 3x \cos 2x \, dx \).

**Solution**

### 6.3 Trigonometric Substitutions

If an integral contains a term of the form \( \sqrt{a^2 - x^2} \), \( \sqrt{a^2 + x^2} \) or \( \sqrt{x^2 - a^2} \), for some \( a > 0 \), you can often evaluate the integral by making a substitution involving a trig function.

First, suppose that an integrand contains a term of the form \( \sqrt{a^2 - x^2} \), for some \( a > 0 \). If we let \( x = a \sin \theta \), where \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), then we can eliminate the square root, as follows. Notice that we now have

\[
\sqrt{a^2 - x^2} = \sqrt{a^2 - (a \sin \theta)^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta
\]

since for \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), \( \cos \theta > 0 \).

**Example 6.17** Evaluate \( \int \frac{\sqrt{16 - x^2}}{x^2} \, dx \).

**Solution**

Next, suppose that an integrand contains a term of the form \( \sqrt{a^2 + x^2} \), for some \( a > 0 \). If we let \( x = a \tan \theta \), where \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), then we can eliminate the square root, as follows. Notice that in this case, we have

\[
\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan \theta)^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = a \sqrt{1 + \tan^2 \theta} = a \sec \theta
\]

since for \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), \( \sec \theta > 0 \).
Example 6.18 Evaluate the integral \( \int \frac{1}{\sqrt{4 + x^2}} \, dx \).

Solution

Finally, suppose that an integrand contains a term of the form \( \sqrt{x^2 - a^2} \), for some \( a > 0 \). If we let \( x = a \sec \theta \), where \( \theta \in \left[ 0, \frac{\pi}{2} \right) \cup \left[ \pi, \frac{3\pi}{2} \right) \), then we can eliminate the square root, as follows. Notice that in this case, we have

\[
\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = a \sqrt{\sec^2 \theta - 1} = a \tan \theta
\]

since for \( \theta \in \left[ 0, \frac{\pi}{2} \right) \cup \left[ \pi, \frac{3\pi}{2} \right) \), \( \tan \theta \geq 0 \).

Example 6.19 Evaluate the integral \( \int \frac{\sqrt{9x^2 - 1}}{x} \, dx \).

Solution

Example 6.20 Evaluate \( \int \frac{x}{\sqrt{x^2 - 6x + 13}} \, dx \).

Solution

We summarize the three trigonometric substitutions presented here in the following table.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta ), ( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} )</td>
<td>( 1 - \sin^2 \theta = \cos^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan \theta ), ( -\frac{\pi}{2} &lt; \theta &lt; \frac{\pi}{2} )</td>
<td>( 1 + \tan^2 \theta = \sec^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta ), ( \theta \in \left[ 0, \frac{\pi}{2} \right) \cup \left[ \pi, \frac{3\pi}{2} \right) )</td>
<td>( \sec^2 \theta - 1 = \tan^2 \theta )</td>
</tr>
</tbody>
</table>

6.4 Integrating Rational Functions by Partial Fractions

In this section we show how to integrate any rational function by expressing it as a sum of simpler fractions, called partial fractions.

Observe that

\[
\frac{3}{x + 2} - \frac{2}{x - 5} = \frac{3(x - 5) - 2(x + 2)}{(x + 2)(x - 5)} = \frac{x - 19}{x^2 - 3x - 10}.
\]

To integrate the function on the right side of this equation, we have

\[
\int \frac{x - 19}{x^2 - 3x - 10} \, dx = \int \left( \frac{3}{x + 2} - \frac{2}{x - 5} \right) \, dx = 3 \ln |x + 2| - 2 \ln |x - 5| + C
\]
To see how the method of partial fractions work in general, let’s consider a rational function

\[ f(x) = \frac{P(x)}{Q(x)} \]  

(6.1)

where \( P \) and \( Q \) are polynomial. It’s possible to express \( f \) as a sum of simpler fractions provided the degree of \( P \) (\( \deg(P) \)) is less than the degree of \( Q \) (\( \deg(Q) \)). Such a rational function is called proper.

If \( f \) is improper, that is, \( \deg(P) \geq \deg(Q) \), then we must take the preliminary step of dividing \( Q \) into \( P \) (by long division) until the remainder \( R(x) \) is obtained such that \( \deg(R) < \deg(Q) \). The division statement is

\[ f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \]

where \( S \) and \( R \) are also polynomial.

**Example 6.21** Find \( \int \frac{x^3 + 2x^2 - 1}{x - 2} \, dx \)

**Solution** .........

The next step is to factor the denominator \( Q(x) \) as far as possible. And the third step is to express the proper rational function \( R(x)/Q(x) \) as a sum of **partial fractions** of the form

\[
\frac{A}{(ax+b)^l} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^l}
\]

**CASE I: The denominator \( Q(x) \) is a product of distinct linear factors.**

This means that we can write

\[ Q(x) = (a_1x+b_1)(a_2x+b_2) \cdots (a_nx+b_n) \]

where no factor is repeated. In this case the partial fraction theorem states that there exist constants \( A_1, A_2, \ldots, A_k \) such that

\[
\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)(a_2x+b_2) \cdots (a_kx+b_k)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k} \]  

(6.2)

These constant can be determined as in the following example.

**Example 6.22** Evaluate \( \int \frac{x - 9}{x^2 + 3x - 10} \, dx \).

**Solution** .........

**Example 6.23** Evaluate \( \int \frac{3x^2 - 7x - 2}{x^3 - x} \, dx \).

**Solution** .........
CASE II: $Q(x)$ is a product of distinct linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated $r$ times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation (6.2), we would use

$$\frac{A_{11}}{a_1x + b_1} + \frac{A_{12}}{(a_1x + b_1)^2} + \cdots + \frac{A_{1r}}{(a_1x + b_1)^r}.$$  \hspace{1cm} (6.3)

For example,

$$\frac{x^2 - 5}{x^2(x + 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} + \frac{D}{(x + 1)^2} + \frac{E}{(x + 1)^3}$$

Example 6.24 Find $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} \, dx$.

Solution ...........

CASE III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations (6.2) and (6.3), the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$  \hspace{1cm} (6.4)

where $A$ and $B$ are constants to be determined. For instance, the fraction given by

$$f(x) = \frac{x}{(x + 2)(x^2 + 1)(x^2 + 2)}$$

has a partial fraction decomposition of the form

$$\frac{x}{(x + 2)(x^2 + 1)(x^2 + 2)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 2}$$

The term given in (6.4) can be integrate by completing the square and using the formula

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 6.25 Evaluate $\int \frac{3x^2 - 4x + 3}{x^3 + x} \, dx$.

Solution ...........

Example 6.26 Evaluate $\int \frac{6x^2 - 3x + 1}{(4x + 1)(x^2 + 1)} \, dx$.

Solution ...........
CASE IV: $Q(x)$ contains a repeated irreducible quadratic factors.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (6.4), the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the term of (6.5) can be integrated by first completing the square.

Example 6.27 Evaluate \(\int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2} \, dx\).

Solution ..........

6.5 Improper Integrals

In defining a definite integral \(\int_a^b f(x) \, dx\) we dealt with a function $f$ defined on a finite interval $[a, b]$ and we assumed that $f$ does not have an infinite discontinuity. In this section we extend the concept of the definite integral to the case where the interval is infinite and also to the case where $f$ has an infinite discontinuity in $[a, b]$. In either case the integral is called an improper integral.

Type I: Infinite Intervals

Definition 6.1

(a) If \(\int_a^t f(x) \, dx\) exists for every number $t \geq a$, then

$$\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx$$

provided this limit exists (as a finite number).

(b) If \(\int_t^b f(x) \, dx\) exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x) \, dx$ and $\int_t^b f(x) \, dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x) \, dx$ and $\int_{-\infty}^a f(x) \, dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx$$

In part (c) any real number $a$ can be used.
Example 6.28  Determine whether the integral
\[ \int_{1}^{\infty} \frac{1}{x} \, dx \]
is convergent or divergent.

Solution ..............

Example 6.29  Evaluate \( \int_{-\infty}^{0} xe^{x} \, dx \).

Solution ..............

Example 6.30  Evaluate \( \int_{-\infty}^{\infty} \frac{1}{1 + x^{2}} \, dx \).

Solution ..............

Example 6.31  What for values of \( p \) is the integral
\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx \]
convergent?

Solution ..............

Type II: Discontinuous Integrands

Definition 6.2

(a) If \( f \) continuous on \([a, b)\) and is discontinuous at \( b \), then
\[ \int_{a}^{b} f(x) \, dx = \lim_{t \to b^{-}} \int_{a}^{t} f(x) \, dx \]
if this limit exists (as a finite number).

(b) If \( f \) continuous on \((a, b]\) and is discontinuous at \( a \), then
\[ \int_{a}^{b} f(x) \, dx = \lim_{t \to a^{+}} \int_{t}^{b} f(x) \, dx \]
if this limit exists (as a finite number).

The improper integrals \( \int_{a}^{b} f(x) \, dx \) is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If \( f \) has a discontinuity at \( c \), where \( a < c < b \), and both \( \int_{a}^{c} f(x) \, dx \) and \( \int_{c}^{b} f(x) \, dx \) are convergent, then we define
\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]
Example 6.32 Find \( \int_{2}^{5} \frac{1}{\sqrt{x-2}} \, dx \).

Solution ...........

Example 6.33 Determine whether the integral

\[
\int_{0}^{\pi/2} \sec x \, dx
\]

converges or diverges.

Solution ...........

Example 6.34 Evaluate \( \int_{0}^{3} \frac{1}{x-1} \, dx \) if possible.

Solution ...........

Example 6.35 Evaluate \( \int_{0}^{1} \ln x \, dx \).

Solution ...........
Chapter 7

Infinite Sequence and Series

7.1 Sequences

A sequence can be thought of as a list of numbers written in a definite order:

\[ a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \]

The number \( a_1 \) is called the first term, \( a_2 \) is the second term, and in general \( a_n \) is the \( n \)th term.

The sequence \( \{a_1, a_2, a_3, \ldots\} \) is also denoted by

\[ \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^\infty \]

Example 7.1 List the first five terms of the following sequence

(a) \( \{2^n\}_{n=1}^\infty = \{2, 4, 8, 16, 32, \ldots\} \)

(b) \( \left\{ \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, \frac{6}{243}, \ldots \right\} \)

(c) \( \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots \right\} \)

Definition 7.1 A sequence \( \{a_n\} \) has the limit \( L \) and we write

\[ \lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty \]

if we can make the terms \( a_n \) as close to \( L \) as we like by taking \( n \) sufficiently large. If \( \lim_{n \to \infty} a_n \) exists, we say that the sequence converges (or is convergent). Otherwise, we say that the sequence diverges (or is divergent).

Limit Laws for Sequences

If \( \{a_n\} \) and \( \{b_n\} \) are convergent sequences and \( c \) is a constant, then

1. \[ \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \]
2. $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$

3. $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n$

4. $\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$

5. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$ if $\lim_{n \to \infty} b_n \neq 0$

6. $\lim_{n \to \infty} c = c$

**Theorem 7.1** If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

**Example 7.2** Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \frac{n}{2n+1}$  (b) $a_n = \frac{(-1)^n}{n}$  (c) $\{7 - 5n\}$

**Solution**

7.2 Infinite Series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^\infty$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

(7.1)

which is called an infinite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^\infty a_n \quad \text{or} \quad \sum a_n$$

To determine whether or not a general series (7.1) has the sum. We consider the partial sum.

$$s_1 = a_1 \quad s_2 = a_1 + a_2 \quad s_3 = a_1 + a_2 + a_3 \quad s_4 = a_1 + a_2 + a_3 + a_4$$

$$\vdots$$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

These partial sum form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n \to \infty} s_n = s$ exist (as a finite number), then we call it the sum of the infinite series $\sum a_n$. 

Definition 7.2 Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote its $n$th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number $s$ is called the **sum** of the series. Otherwise, the series is called **divergent**.

Example 7.3 Determine whether the series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

is convergent or divergent. If it is convergent, find its sum.

Solution Consider the partial sum:

$$s_1 = \frac{1}{2}$$
$$s_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4} = 1 - \frac{1}{2^2}$$
$$s_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8} = 1 - \frac{1}{2^3}$$
$$s_4 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{15}{16} = 1 - \frac{1}{2^4}$$
$$\vdots$$
$$s_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Therefore

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1,$$

that is, the sequence $\{s_n\}$ converges to 1. Hence the series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is convergent and its sum is 1. •

Definition 7.3 The **geometric series** is the series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0$$

and the number $r$ is called the **ratio** for the series.
Here are some examples:

- \[ 1 + 2 + 4 + 8 + \cdots + 2^{n-1} + \cdots; \quad a = 1, \quad r = 2 \]
- \[ \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{n+1} \frac{1}{2^n} + \cdots; \quad a = \frac{1}{2}, \quad r = -\frac{1}{2} \]
- \[ \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^n} + \cdots; \quad a = \frac{3}{10}, \quad r = \frac{1}{10} \]

**Theorem 7.2**  The geometric series

\[
\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots
\]

is convergent if \(|r| < 1\) and its sum is

\[
s = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1
\]

If \(|r| \geq 1\), the geometric series is divergent.

**Example 7.4**  Find the sum of \[ \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots. \]

**Solution** …………

**Example 7.5**  Is the series \( \sum_{n=1}^{\infty} 2^{n}3^{1-n} \) convergent or divergent?

**Solution** …………

**Example 7.6**  Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) is convergent, and find its sum.

**Solution**  The \( n \)th partial sum of the series is

\[
s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}
\]

We can simplify this expression if we use the partial fraction decomposition

\[
\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}
\]

Thus, we have

\[
s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}
\]
and so
\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - 0 = 1 \]

Therefore, the given series is convergent and
\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \checkmark \]

**Definition 7.4** *The harmonic series* is the divergent series of the form
\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

**Theorem 7.3** If the series \( \sum_{n=1}^{\infty} a_n \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \).

**Note.** The converse of Theorem 7.3 is not true in general. If \( \lim_{n \to \infty} a_n = 0 \), we cannot conclude that \( \sum a_n \) is convergent. Observe that for the harmonic series \( \sum \frac{1}{n} \) we have \( a_n = 1/n \to 0 \) as \( n \to \infty \), but the harmonic series is divergent.

### 7.3 Convergence Tests

In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

#### 7.3.1 The Divergence Test

**Theorem 7.4** (The Divergence Test)

(a) If \( \lim_{n \to \infty} a_n \) does not exist or if \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

(b) If \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) may either converge or diverge.

**Example 7.7** Show that the series \( \sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4} \) diverges.

**Solution** ....

**Theorem 7.5** If \( \sum a_n \) and \( \sum b_n \) are convergent series, then so are the series \( \sum ca_n \) (where \( c \) is a constant), \( \sum (a_n + b_n) \), and \( \sum (a_n - b_n) \), and

(i) \( \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n \)
Example 7.8 Find the sum of the series

\[ \sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right) \].

Solution . . . . . . . .

Theorem 7.6 The p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

Example 7.9

(a) The series \( \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \) is convergent because it is a p-series with \( p = 3 > 1 \).

(b) The series \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \) is divergent because it is a p-series with \( p = \frac{1}{3} < 1 \).

7.3.2 The Limit Comparison Test

Theorem 7.7 (The Limit Comparison Test) Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. If

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = c \]

where \( c \) is a finite number and \( c > 0 \), then either both series converge or both diverge.

Example 7.10 Determine whether the series converges or diverges.

(a) \( \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5} + n^3} \) 

(b) \( \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \)

Solution . . . . . . . .

7.3.3 The Ratio Test

Theorem 7.8 (The Ratio Test) Let \( \sum a_n \) be a series with positive terms and suppose that

\[ \rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \]

(i) If \( \rho < 1 \), then the series converges.
(ii) If $\rho > 1$ or $\rho = \infty$, then the series diverges.

(iii) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Example 7.11 Determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$  

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{4^n}$

Solution . . . . . . . . .

7.3.4 The Root Test

Theorem 7.9 (The Root Test) Let $\sum a_n$ be a series with positive terms and suppose that

$$\rho = \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} (a_n)^{1/n}$$

(i) If $\rho < 1$, the series converges.

(ii) If $\rho > 1$ or $\rho = \infty$, then the series diverges.

(iii) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Example 7.12 Determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$  

(b) $\sum_{n=2}^{\infty} \left(\frac{1}{\ln n}\right)^n$

Solution . . . . . . . . .

7.4 Taylor and Maclaurin Series; Power Series

Taylor and Maclaurin Series

Definition 7.5 If $f$ has derivatives for all order at $a$, then we call the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \cdots$$

the Taylor series of the function $f$ at $a$ (or about $a$ or centered at $a$). In the special case where $a = 0$, this series becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

in which case we call it the Maclaurin series.
Example 7.13  Find the Maclaurin series for

(a) \( f(x) = e^x \)  \hspace{1cm}  (b) \( f(x) = \sin x \)

Solution ...........

Example 7.14  Find the Taylor series of \( f(x) = \frac{1}{x} \) centered at 1.

Solution ...........

♠♠♠✠✠♣♣♣⋆⋆⋆♦♦♦

Good Luck  ♠♠♠♣♣♣✠✠♠♠♠