Chapter 7

Topics in Vector Calculus

7.1 Vector Fields

**Definition 7.1**

A vector field in 2-space is a function that maps each point \((x, y)\) to a unique vector

\[ \mathbf{F}(x, y) = f(x, y) \mathbf{i} + g(x, y) \mathbf{j} = \langle f(x, y), g(x, y) \rangle. \]

The functions \(f\) and \(g\) are called the components of the vector field \(\mathbf{F}\).

Similarly, a vector field in 3-space is a function that maps each point \((x, y, z)\) to a unique vector

\[ \mathbf{F}(x, y, z) = f(x, y, z) \mathbf{i} + g(x, y, z) \mathbf{j} + h(x, y, z) \mathbf{k} = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle. \]

The functions \(f\), \(g\), and \(h\) are called the components of the vector field \(\mathbf{F}\).

Graphical Representations of Vector Fields

A vector field in 2-space can be pictured geometrically by drawing representative field vectors \(\mathbf{F}(x, y)\) at some well-chosen points in the \(xy\)-plane.
Describe the vector field \( \vec{F} \) if \( \vec{F}(x, y) = -yi + xj \).

**Solution**
The vectors \( \vec{F}(x, y) \) associated with several points \((x, y)\) are listed in the following table and sketched in Figure below.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(\vec{F}(x, y))</th>
<th>((x, y))</th>
<th>(\vec{F}(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1))</td>
<td>(-i + j)</td>
<td>((1, 3))</td>
<td>(-3i + j)</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>(-i - j)</td>
<td>((-3, 1))</td>
<td>(-i - 3j)</td>
</tr>
<tr>
<td>((-1, -1))</td>
<td>(i - j)</td>
<td>((-1, -3))</td>
<td>(3i - j)</td>
</tr>
<tr>
<td>((1, -1))</td>
<td>(i + j)</td>
<td>((3, -1))</td>
<td>(i + 3j)</td>
</tr>
</tbody>
</table>

**Gradient Fields**
An important class of vector fields arise from the process of finding gradients. Recall that if \( \phi \) is a function of three variables, then the gradient of \( \phi \) is defined as

\[
\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle
\]

This formula defines a vector field in 3-space called the gradient field of \( \phi \).

Similarly, the gradient field of a function of two variables defines a gradient field in 2-space.

At each point in a gradient field where the gradient is nonzero, the vector points in the direction in which the rate of increase of \( \phi \) is maximum.

**Example 7.2**
Find the gradient vector field of the following functions.

(a) \( \phi(x, y) = x^2 \sin(5y) \)

(b) \( \phi(x, y, z) = ze^{-xy} \)

**Solution**
Conservative Fields and Potential Functions

If \( \vec{F} \) is an arbitrary vector field in 2-space or 3-space, we can ask whether it is the gradient field of some function \( \phi \), and if so, how we can find \( \phi \). This is an important problem in various applications. There is some terminology for such fields that we will introduce now.

**Definition 7.2**

A vector field \( \vec{F} \) in 2-space or 3-space is said to be **conservative** in a region if it is the gradient field for some function \( \phi \) in that region; that is, if

\[
\vec{F} = \nabla \phi
\]

The function \( \phi \) is called a **potential function** for \( \vec{F} \) in the region.

In a later section we will discuss methods for finding potential functions for conservative vector fields.

Divergence and Curl

We will now define two important operations on vector fields in 3-space—the **divergence** and the **curl** of the field. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows toward or away from a point and the curl relates to the rotational properties of the fluid at a point.

**Definition 7.3**

If \( \vec{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k} \), then we define the **divergence of \( \vec{F} \)**, written \( \text{div} \vec{F} \), to be the function given by

\[
\text{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z},
\]

and the **curl of \( \vec{F} \)**, written \( \text{curl} \vec{F} \), to be the vector field given by

\[
\text{curl} \vec{F} = \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}.
\]

Note that \( \text{div} \vec{F} \) has scalar values, whereas \( \text{curl} \vec{F} \) has vector values (i.e., \( \text{curl} \vec{F} \) is itself a vector field). Moreover, the curl can be expressed in the determinant form

\[
\text{curl} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f & g & h
\end{vmatrix}
\]

(7.3)
Find the divergence and the curl of the vector field
\[
\vec{F}(x, y, z) = x^2 y \vec{i} + 2y^3 z \vec{j} + 3z \vec{k}.
\]

Solution

The \( \nabla \) Operator

Thus far, the symbol \( \nabla \) that appears in the gradient expression \( \nabla \phi \) has not been given a meaning of its own. However, it is often convenient to view \( \nabla \) as an operator
\[
\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \quad (7.4)
\]
which when applied to \( \phi(x, y, z) \) produces the gradient
\[
\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}.
\]
We call (7.4) the del operator. This is analogous to the derivative operator \( d/dx \), which when applied to \( f(x) \) produces the derivative \( f'(x) \).

The del operator allows us to express the divergence of a vector field
\[
\vec{F}(x, y, z) = f(x, y, z) \vec{i} + g(x, y, z) \vec{j} + h(x, y, z) \vec{k} \quad (7.5)
\]
in dot product notation as
\[
\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \quad (7.6)
\]
and the curl of this field in cross-product notation as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$ (7.7)

Exercise 7.1

1 – 2 Confirm that $\phi$ is a potential function for $\vec{F}(\vec{r})$ on some region.

1. (a) $\phi(x, y) = \tan^{-1} xy$
   $$\vec{F}(x, y) = \frac{y}{1 + x^2y^2} \hat{i} + \frac{x}{1 + x^2y^2} \hat{j}$$
   (b) $\phi(x, y, z) = x^2 - 3y^2 + 4z^2$
   $$\vec{F}(x, y, z) = 2x\hat{i} - 6y\hat{j} + 8z\hat{k}$$

2. (a) $\phi(x, y) = 2y^2 + 3x^2y - xy^3$
   $$\vec{F}(x, y) = (6xy - y^3)\hat{i} + (4y + 3x^2 - 3xy^2)\hat{j}$$
   (b) $\phi(x, y, z) = x\sin z + y\sin x + z\sin y$
   $$\vec{F}(x, y, z) = (\sin z + y\cos x)\hat{i} + (\sin x + z\cos y)\hat{j} + (\sin y + x\cos z)\hat{k}$$

3 – 8 Find div $\vec{F}$ and curl $\vec{F}$.

3. $\vec{F}(x, y, z) = x^2\hat{i} - 2\hat{j} + yz\hat{k}$
4. $\vec{F}(x, y, z) = xz^3\hat{i} + 2y^4x^2\hat{j} + 5z^2y\hat{k}$

5. $\vec{F}(x, y, z) = 7y^3z^2\hat{i} - 8x^2z^5\hat{j} - 3xy^4\hat{k}$
6. $\vec{F}(x, y, z) = e^{xy}\hat{i} - \cos y\hat{j} + \sin^2 z\hat{k}$

7. $\vec{F}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\hat{i} + y\hat{j} + z\hat{k})$
8. $\vec{F}(x, y, z) = \ln x\hat{i} + e^{yz}\hat{j} + \tan^{-1}(z/x)\hat{k}$

9 – 10 Find $\nabla \cdot (\vec{F} \times \vec{G})$.

9. $\vec{F}(x, y, z) = 2x\hat{i} + \hat{j} + 4y\hat{k}$
   $$\vec{G}(x, y, z) = x\hat{i} + y\hat{j} - z\hat{k}$$

10. $\vec{F}(x, y, z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$
    $$\vec{G}(x, y, z) = xy\hat{i} + xyz\hat{k}$$

11 – 12 Find $\nabla \cdot (\nabla \times \vec{F})$.

11. $\vec{F}(x, y, z) = \sin x\hat{i} + \cos(x - y)\hat{j} + z\hat{k}$
    12. $\vec{F}(x, y, z) = e^{xz}\hat{i} + 3xe^{y}\hat{j} - e^{yz}\hat{k}$

13 – 14 Find $\nabla \times (\nabla \times \vec{F})$.

13. $\vec{F}(x, y, z) = xy\hat{j} + yz\hat{k}$
    14. $\vec{F}(x, y, z) = y^2x\hat{i} - 3yz\hat{j} + xy\hat{k}$
Answers to Exercise 7.1

1. (a) all \( x, y \)  
   (b) all \( x, y \)

3. \( \text{div} \vec{F} = 2x + y \), \( \text{curl} \vec{F} = z \vec{i} \)

5. \( \text{div} \vec{F} = 0 \), \( \text{curl} \vec{F} = 2x^2 + z^2 \vec{i} \)

7. \( \text{div} \vec{F} = \frac{2}{\sqrt{x^2 + y^2 + z^2}} \), \( \text{curl} \vec{F} = 0 \)

9. 4

11. 0

13. \((1 + y) \vec{i} + x \vec{j}\)

7.2 Line Integral

Line Integral

We have so far integrated “over” intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or “along” a curve—“line integrals” are really “curve integrals”.

Let’s start with the curve \( C \) in the \( xy \)-plane. We will assume that the curve is smooth and is given by the parametric equations,

\[
    x = x(t), \quad y = y(t) \quad (a \leq t \leq b)
\]

We will often want to write the parameterization of the curve as a vector function. In this case the curve is given by,

\[
    \vec{r}(t) = x(t) \vec{i} + y(t) \vec{j} \quad (a \leq t \leq b)
\]

The curve is called smooth if \( \vec{r}'(t) \) is continuous and \( \vec{r}'(t) \neq \vec{0} \) for all \( t \).

**Definition 7.4**

For a real-valued function \( f(x, y) \) and a curve \( C \) in the \( xy \)-plane, parametrized by \( x = x(t), \ y = y(t), \ a \leq t \leq b \), the line integral of \( f(x, y) \) along \( C \) with respect to arc length \( s \) is

\[
    \int_C f(x, y) \, ds = \int_{a}^{b} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \quad (7.8)
\]

If we use the vector form of the parameterization we can simplify the notation up somewhat by noticing that

\[
    \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \|\vec{r}'(t)\|
\]

where \( \|\vec{r}'(t)\| \) is the magnitude or norm of \( \vec{r}'(t) \). Using this notation the line integral becomes,

\[
    \int_C f(x, y) \, ds = \int_{a}^{b} f(x(t), y(t)) \|\vec{r}'(t)\| \, dt \quad (7.9)
\]
Example 7.4

Evaluate \( \int_C xy^4 \, ds \) where \( C \) is the right half of the circle, \( x^2 + y^2 = 4 \).

Solution

Example 7.5

Using the given parametrization, evaluate the line integral \( \int_C (1 + xy^2) \, ds \).

(a) \( C : \vec{r}(t) = t\vec{i} + 2t\vec{j} \quad (0 \leq t \leq 1) \)

(b) \( C : \vec{r}(t) = (1 - t)\vec{i} + (2 - 2t)\vec{j} \quad (0 \leq t \leq 1) \)

Solution
Note that the integrals in part (a) and (b) of Example 7.5 agree, even though the corresponding parametrizations of $C$ have opposite orientations. This illustrates the important result that the value of a line integral of $f$ along $C$ does not depend on an orientation of $C$. That is,

$$\int_C f(x, y) \, ds = \int_C f(x, y) \, ds$$  \hfill (7.10)

We can do line integrals over three dimensional curves as well. Let’s suppose that the three-dimensional curve $C$ is given by the parameterization,

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \leq t \leq b)$$

then the line integral is given by,

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$  \hfill (7.11)

Note that often when dealing with three-dimensional space the parameterization will be given as a vector function.

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad (a \leq t \leq b)$$

Also notice that, as with two-dimensional curves, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \|\vec{r}'(t)\|,$$

and the line integral can again be written as

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t))\|\vec{r}'(t)\| \, dt$$  \hfill (7.12)

Note:

- If $C$ is a curve in 3-space that model as a thin wire, and if $f(x, y, z)$ is a linear density function of the wire, then the mass $M$ of the wire is given by

$$M = \int_C f(x, y, z) \, ds$$  \hfill (7.13)

- If $C$ is a smooth curve of arc length $L$, and $f$ is identically 1, then

$$\int_C ds = L$$  \hfill (7.14)
Evaluate the line integral \( \int_C (xy - z^3) \, ds \) from \((1, 0, 0)\) to \((-1, 0, \pi)\) along the helix \(C\) that is represented by parametric equations

\[
x = \cos t, \quad y = \sin t, \quad z = t \quad (0 \leq t \leq \pi)
\]

Solution
Line integrals with respect to $x$, $y$, and $z$

In the previous topic we looked at line integrals with respect to arc length. In this topic we want to look at line integrals with respect to $x$, $y$, or $z$.

As with the last topic we will start with a two-dimensional curve $C$ with parameterization,

$$x = x(t), \quad y = y(t) \quad (a \leq t \leq b)$$

The **line integral of $f$ along $C$ with respect to $x$** is

$$\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) \, x'(t) \, dt \quad (7.15)$$

The **line integral of $f$ along $C$ with respect to $y$** is

$$\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) \, y'(t) \, dt \quad (7.16)$$

Frequently, the line integrals with respect to $x$ and $y$ occur in combination, in which case we will dispense with one of the integral signs and write

$$\int_C f(x, y) \, dx + g(x, y) \, dy = \int_C f(x, y) \, dx + \int_C g(x, y) \, dy \quad (7.17)$$

**Example 7.7**

Evaluate $\int_C yx^2 \, dx + \sin(\pi y) \, dy$ where $C$ is the line segment from $(0, 2)$ to $(1, 4)$.

**Solution**
If \( C \) is a smooth oriented curve, we will let \(-C\) denote the oriented curve consisting of the same point as \( C \) but with the opposite orientation.

We then have

\[
\int_C f(x, y) \, dx = -\int_C f(x, y) \, dx
\]

and

\[
\int_C g(x, y) \, dy = -\int_C g(x, y) \, dy
\]

With the combined form of these two integrals we get

\[
\int_C f(x, y) \, dx + g(x, y) \, dy = -\int_C f(x, y) \, dx + g(x, y) \, dy
\]

If \( C \) is a smooth curve in 3 space, we can have line integrals of \( f \) along \( C \) with respect to \( x \), \( y \), and \( z \).

\[
\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t)) \, x'(t) \, dt \\
\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t)) \, y'(t) \, dt \\
\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) \, z'(t) \, dt
\]

where the curve \( C \) is parameterized by

\[ x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \leq t \leq b) \]

As with the two-dimensional version these three will often occur together so the shorthand we’ll be using here is,

\[
\int_C f \, dx + g \, dy + h \, dz = \int_C f(x, y, z) \, dx + \int_C g(x, y, z) \, dy + \int_C h(x, y, z) \, dz
\]
Example 7.8

Evaluate \( \int_C y \, dx + x \, dy + z \, dz \) where \( C \) is given by \( x = \cos t, \ y = \sin t, \ z = t^2, \ 0 \leq t \leq 2\pi \).

Solution

Line Integrals of Vector Fields

In the previous two topics we looked at line integrals of functions. In this topic we are going to evaluate line integrals of vector fields. We’ll start with the vector field

\[
\vec{F}(x, y, z) = f(x, y, z) \vec{i} + g(x, y, z) \vec{j} + h(x, y, z) \vec{k},
\]

and the three-dimensional smooth oriented curve \( C \) given by

\[
\vec{r}(t) = x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k}.
\]

The line integral of \( \vec{F} \) along \( C \) is

\[
\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \quad \text{(7.18)}
\]

Note that \( \vec{F}(\vec{r}(t)) \) is a shorthand for

\[
\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t)).
\]
Evaluate $\int_{C} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = 8x^2yz \mathbf{i} + 5z \mathbf{j} - 4xy \mathbf{k}$ and where $C$ is the curve given by $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$, $0 \leq t \leq 1$.

Solution

Let’s now take a closer look at a nice relationship between line integrals of vector fields and line integrals with respect to $x$, $y$, and $z$.

Given the vector field $\mathbf{F}(x, y, z) = f(x, y, z) \mathbf{i} + g(x, y, z) \mathbf{j} + h(x, y, z) \mathbf{k}$ and the curve $C$ parameterized by $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$, $a \leq t \leq b$, the line integral is

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} (f(x, y, z)x'(t) + g(x, y, z)y'(t) + h(x, y, z)z'(t)) \, dt$$

$$= \int_{a}^{b} f(x, y, z)x'(t) \, dt + \int_{a}^{b} g(x, y, z)y'(t) \, dt + \int_{a}^{b} h(x, y, z)z'(t) \, dt$$

$$= \int_{C} f(x, y, z) \, dx + \int_{C} g(x, y, z) \, dy + \int_{C} h(x, y, z) \, dz$$

So, we see that

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz$$
Work as a Line Integral

In Section 3.3 we took the concept of work a step further by defining the work $W$ performed by a constant force $\vec{F}$ moving a particle in a straight line from point $P$ to point $Q$. We defined the work to be

$$ W = \vec{F} \cdot \vec{PQ}. $$

Our next goal is to define a more general concept of work—the work performed by a variable force acting on a particle that moves along a curve path in 2-space or 3-space.

**Definition 7.5**

Suppose that under the influence of a continuous force field $\vec{F}$ a particle moves along a smooth curve $C$ and that $C$ is oriented in the direction of motion of the particle. Then the work performed by the force field on the particle is

$$ W = \int_C \vec{F} \cdot d\vec{r}. $$

Line Integrals Along Piecewise Smooth Curves

Thus far, we have only considered line integrals along smooth curves. However, the notion of a line integral can be extended to curves formed from finitely many smooth curves $C_1, C_2, \ldots, C_n$ joined end to end. Such a curve is called piecewise smooth.

We define a line integral along a piecewise smooth curve $C$ to be the sum of the integrals along the sections:

$$ \int_C = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n} $$
Example 7.10

Evaluate \( \int_C x^2y \, dx + x \, dy \) along the curve \( C \) shown in the figure below.

Solution
Exercise 7.2

1. Let \( C \) be the line segment from \((0,0)\) to \((0,1)\). In each part, evaluate the line integral along \( C \) by inspection, and explain your reasoning.
   \[
   (a) \int_C ds \\
   (b) \int_C \sin xy\, dy
   \]

2. Let \( C \) be the line segment from \((0,2)\) to \((0,4)\). In each part, evaluate the line integral along \( C \) by inspection, and explain your reasoning.
   \[
   (a) \int_C ds \\
   (b) \int_C e^{xy}\, dx
   \]

3. Let \( C \) be the curve represented by the equations
   \[
   x = 2t, \quad y = t^2 \quad (0 \leq t \leq 1)
   \]
   In each part, evaluate the line integral along \( C \).
   \[
   (a) \int_C (x - \sqrt{y})\, ds \\
   (b) \int_C (x - \sqrt{y})\, dx \\
   (c) \int_C (x - \sqrt{y})\, dy
   \]

4. Let \( C \) be the curve represented by the equations
   \[
   x = t, \quad y = 3t^2, \quad z = 6t^3 \quad (0 \leq t \leq 1)
   \]
   In each part, evaluate the line integral along \( C \).
   \[
   (a) \int_C xyz^2\, ds \\
   (b) \int_C xyz^2\, dx \\
   (c) \int_C xyz^2\, dy \\
   (d) \int_C xyz^2\, dz
   \]

5. In each part, evaluate the integral \( \int_C (3x + 2y)\, dx + (2x - y)\, dy \) along the stated curve.
   (a) The line segment from \((0,0)\) to \((1,1)\).
   (b) The parabolic arc \( y = x^2 \) from \((0,0)\) to \((1,1)\).
   (c) The curve \( y = \sin(\pi x/2) \) from \((0,0)\) to \((1,1)\).
   (d) The curve \( x = y^3 \) from \((0,0)\) to \((1,1)\).

6. In each part, evaluate the integral \( \int_C y\, dx + z\, dy - x\, dz \) along the stated curve.
   (a) The line segment from \((0,0,0)\) to \((1,1,1)\).
   (b) The twisted cubic \( x = t, \ y = t^2, \ z = t^3 \) from \((0,0,0)\) to \((1,1,1)\).
   (c) The helix \( x = \cos pt, \ y = \sin \pi t, \ z = t \) from \((1,0,0)\) to \((-1,0,1)\).
7 – 10. Evaluate the line integral with respect to \( s \) along the curve \( C \).

7. \[ \int_C \frac{1}{1+x} \, ds, \quad C: \vec{r}(t) = t\vec{i} + \frac{2}{3} t^{3/2} \vec{j} \quad (0 \leq t \leq 3) \]

8. \[ \int_C \frac{x}{1+y^2} \, ds, \quad C: x = 1 + 2t, \ y = t \quad (0 \leq t \leq 1) \]

9. \[ \int_C 3x^2yz \, ds, \quad C: x = t, \ y = t^2, \ z = \frac{2}{3} t^3 \quad (0 \leq t \leq 1) \]

10. \[ \int_C \frac{e^{-z}}{x^2 + y^2} \, ds, \quad C: \vec{r}(t) = 2 \cos t \vec{i} + 2 \sin t \vec{j} + t \vec{k} \quad (0 \leq t \leq 2\pi) \]

11 – 18. Evaluate the line integral along the curve \( C \).

11. \[ \int_C (x + 2y) \, dx + (x - y) \, dy, \quad C: x = 2 \cos t, \ y = 4 \sin t \quad (0 \leq t \leq \pi/4) \]

12. \[ \int_C (x^2 - y^2) \, dx + x \, dy, \quad C: x = t^{2/3}, \ y = t \quad (-1 \leq t \leq 1) \]

13. \[ \int_C -y \, dx + x \, dy, \quad C: y^2 = 3x \text{ from } (3, 3) \text{ to } (0, 0). \]

14. \[ \int_C (y - x) \, dx + x^2 y \, dy, \quad C: y^2 = x^3 \text{ from } (1, -1) \text{ to } (1, 1). \]

15. \[ \int_C (x^2 + y^2) \, dx - x \, dy, \quad C: x^2 + y^2 = 1, \text{ counterclockwise from } (1, 0) \text{ to } (0, 1). \]

16. \[ \int_C (y - x) \, dx + xy \, dy, \quad C: \text{ the line segment from } (3, 4) \text{ to } (2, 1). \]

17. \[ \int_C yz \, dx - xz \, dy + xy \, dz, \quad C: x = e^t, \ y = e^{3t}, \ z = e^{-t} \quad (0 \leq t \leq 1) \]

18. \[ \int_C x^2 \, dx + xy \, dy + z^2 \, dz, \quad C: x = \sin t, \ y = \cos t, \ z = t^2 \quad (0 \leq t \leq \pi/2) \]
19 – 20 Evaluate $\int_C y \, dx - x \, dy$ along the curve $C$ shown in the figure.

19. (a)

![Diagram for 19(a)]

(b)

![Diagram for 19(b)]

20. (a)

![Diagram for 20(a)]

(b)

![Diagram for 20(b)]

21 – 24 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $C$.

21. \( \vec{F}(x, y) = x^2 \vec{i} + xy \vec{j}, \quad C : \vec{r}(t) = 2 \cos t \vec{i} + 2 \sin t \vec{j} \quad (0 \leq t \leq \pi) \)

22. \( \vec{F}(x, y) = x^2 y \vec{i} + 4 \vec{j}, \quad C : \vec{r}(t) = e^t \vec{i} + e^{-t} \vec{j} \quad (0 \leq t \leq 1) \)
23. \( \vec{F}(x, y) = (x^2 + y^2)^{-3/2}(x\vec{i} + y\vec{j}) \), \( C : \vec{r}(t) = e^t\sin t\vec{i} + e^t\cos t\vec{j} \) \( (0 \leq t \leq 1) \)

24. \( \vec{F}(x, y, z) = z\vec{i} + x\vec{j} + y\vec{k} \), \( C : \vec{r}(t) = \sin t\vec{i} + 3\sin t\vec{j} + \sin^2 t\vec{k} \) \( (0 \leq t \leq \pi/2) \)

25 – 28 Find the work done by the force field \( \vec{F} \) on a particle that moves along the curve \( C \).

25. \( \vec{F}(x, y) = xy\vec{i} + x^2\vec{j} \), \( C : x = y^2 \) from \((0, 0)\) to \((1, 1)\).

26. \( \vec{F}(x, y) = (x^2 + xy)\vec{i} + (y - x^2y)\vec{j} \), \( C : x = t, y = 1/t \) \( (1 \leq t \leq 3) \)

27. \( \vec{F}(x, y, z) = xy\vec{i} + yz\vec{j} + xz\vec{k} \), \( C : \vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k} \) \( (0 \leq t \leq 1) \)

28. \( \vec{F}(x, y, z) = (x + y)\vec{i} + xy\vec{j} - z^2\vec{k} \)

\( C : \) along line segments from \((0, 0, 0)\) to \((1, 3, 1)\) to \((2, -1, 4)\).

28 – 29 Find the work done by the force field
\[
\vec{F}(x, y) = \frac{1}{x^2 + y^2} \vec{i} + \frac{4}{x^2 + y^2} \vec{j}
\]
on a particle that moves along the curve \( C \) show in the figure.

28.

![Diagram](image)

29.

Answers to Exercise 7.2

1. (a) 1  (b) 0  (c) \( \frac{4\sqrt{2} - 2}{3} \)  (d) \( \frac{2}{3} \)

2. 9  \( \frac{13}{20} \) 11  \( 1 - \pi \) 13  3  15  \( -1 - \frac{\pi}{4} \) 17  \( 1 - e^3 \) 19. (a) \( -1 \)  (b) \( -2 \)

21. 0 23. \( 1 - e^{-1} \) 25. \( \frac{3}{5} \) 27. \( \frac{27}{28} \) 28. \( \frac{3}{4} \)
7.3 Independence of Path; Conservative Vector Fields

A piecewise smooth curve with endpoints $P$ and $Q$ is sometimes called a \textit{path} from $P$ to $Q$. We now obtain conditions under which a line integral is \textit{independent of path} in a region in the sense that if $P$ and $Q$ are arbitrary points, then the same value is obtained for every path in the region form $P$ to $Q$.

Throughout this section we assume that all regions $D$ are \textit{connected}; that is, any two points in $D$ can be jointed by a piecewise smooth curve that lies entirely in $D$. Stated informally, $D$ is connected if it does not consist of two or more separate pieces.

If $\vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j}$ is continuous on an open connected region $D$, then the line integral \( \int_{C} \vec{F} \cdot d\vec{r} \) is independent of path if and only if $\vec{F}$ is conservative; that is, $\vec{F}(x, y) = \nabla \phi(x, y)$ for some potential function $\phi$.

Recall from the Fundamental Theorem of Calculus that if $F$ is an antiderivative of $f$, then
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]
It appears that there is an analog of this theorem for line integrals.
Let \( \vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j} \) be continuous on an open connected region \( D \), and let \( C \) be a piecewise smooth curve in \( D \) with endpoints \((x_0, y_0)\) and \((x_1, y_1)\). If \( \vec{F}(x, y) = \nabla \phi(x, y) \), then

\[
\int_C \vec{F}(x, y) \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{7.20}
\]

or, equivalently,

\[
\int_C \nabla \phi \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{7.21}
\]

Stated informally, this theorem shows that the values of a line integral of a conservative vector field along a piecewise smooth path is independent of the path; that is, the value of the integral depends on the endpoints and not on the actual path. Accordingly, for line integrals along paths in conservative vector fields, it is common to express (7.20) and (7.21) as

\[
\int_{(x_0, y_0)}^{(x_1, y_1)} \vec{F} \cdot d\vec{r} = \int_{(x_0, y_0)}^{(x_1, y_1)} \nabla \phi \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{7.22}
\]

If a line integral \( \int_C \vec{F} \cdot d\vec{r} \) is independent of path, then using Theorem 7.2 with \((x_0, y_0) = (x_1, y_1)\) we see that

\[
\int_C \vec{F} \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0) = 0
\]

for every simple closed curve \( C \). Note that a curve is simple if it does not intersect itself.
Example 7.11

(a) Confirm that the force field $\vec{F}(x, y) = yi + xj$ is conservative by showing that $\vec{F}(x, y)$ is the gradient of $\phi(x, y) = xy$.

(b) Use the Fundamental Theorem of Line Integrals to evaluate $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r}$.

Solution

The following result is an application to conservative vector fields.

**Theorem 7.3**

If $\vec{F}$ is a conservative vector field in 2-space, then the work done by $\vec{F}$ along any path $C$ from $P(x_0, y_0)$ to $Q(x_1, y_1)$ is equal to the difference in potentials between $P$ and $Q$.

The following theorem is the primary tool for determining whether a vector field in 2-space is conservative.

**Theorem 7.4 (Conservative Field Test)**

If $f(x, y)$ and $g(x, y)$ have continuous first derivatives on simply connected region $D$, then the vector field

$$\vec{F}(x, y) = f(x, y)i + g(x, y)j$$

is conservative on $D$ if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad (7.23)$$

at each point in $D$. 
It is not hard to see why (7.23) must hold if $\mathbf{F}$ is conservative. For this purpose suppose that $\mathbf{F} = \nabla \phi$, in which case we can express the functions $f$ and $g$ as

$$\frac{\partial \phi}{\partial x} = f \quad \text{and} \quad \frac{\partial \phi}{\partial y} = g \quad (7.24)$$

Thus,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial^2 \phi}{\partial y \partial x} \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial^2 \phi}{\partial x \partial y}$$

But the mixed partial derivatives in these equations are equal, so (7.23) follows.

**Example 7.12**

Determine whether the vector field $\mathbf{F}(x, y) = x^2 y \mathbf{i} + 3xy^2 \mathbf{j}$ is conservative.

**Solution**

**Example 7.13**

Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left( e^{3y} - y^2 \sin x \right) dx + \left( 3xe^{3y} + 2y \cos x \right) dy$$

is independent of path in a simply connected region.

**Solution**

Once it is established that a vector field is conservative, a potential function for the field can be obtained by first integrating either of the equations in (7.24). This is illustrated in the following example.
Example 7.14

Let \( \vec{F}(x, y) = 2xy^3 \vec{i} + (1 + 3x^2 y^2) \vec{j} \).

(a) Show that \( \vec{F} \) is a conservative vector field on the entire \( xy \)-plane.

(b) Find \( \phi \) by first integrating \( \partial \phi / \partial x \).

(c) Find \( \phi \) by first integrating \( \partial \phi / \partial y \).

Solution
Use the potential function obtained in Example 7.14 to evaluate the integral

\[ \int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy \]

Solution

**Conservative Vector Fields in 3-Space**

All of the result in this section have analogs in 3-space: The Theorem 7.2 and 7.1 can be extended to vector fields in 3-space simply by adding a third variable and modifying the hypotheses appropriately. For example, in 3-space, Formula (7.20) becomes

\[ \int_C \vec{F}(x, y, z) \cdot d\vec{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) \]  

\[ (7.25) \]

Theorem 7.4 can also be extended to vector fields in 3-space. We can show that if \( \vec{F}(x, y, z) = f(x, y, z)\vec{i} + g(x, y, z)\vec{j} + h(x, y, z)\vec{k} \) is a conservative field, then

\[ \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \]  

\[ (7.26) \]

that is, \( \text{curl} \, \vec{F} = \vec{0} \). Conversely, a vector field satisfying these conditions on a suitably restricted region is conservative on the region if \( f, g, \) and \( h \) are continuous and have continuous first partial derivatives in the region.

**Exercise 7.3**

1 - 6 Determine whether \( \vec{F} \) is a conservative vector field. If so, find a potential function for it.

1. \( \vec{F}(x, y) = x\vec{i} + y\vec{j} \)
2. \( \vec{F}(x, y) = 3y^2\vec{i} + 6xy\vec{j} \)
3. \( \vec{F}(x, y) = x^2y\vec{i} + 5xy^2\vec{j} \)
4. \( \vec{F}(x, y) = e^x \cos y\vec{i} - e^x \sin y\vec{j} \)
5. \( \vec{F}(x, y) = x \ln y\vec{i} + y \ln x\vec{j} \)
6. \( \vec{F}(x, y) = (\cos y + y \cos x)\vec{i} + (\sin x - x \sin y)\vec{j} \)
7. (a) Show that the line integral \( \int_C y^2 \, dx + 2xy \, dy \) is independent of the path.

(b) Evaluate the integral in part (a) along the line segment from \((-1, 2)\) to \((1, 3)\).

(c) Evaluate the integral \( \int_{(-1,2)}^{(1,3)} y^2 \, dx + 2xy \, dy \) using Theorem 16.1, and confirm that the value is the same as that obtained in part (b).

8. (a) Show that the line integral \( \int_C y \sin x \, dx - \cos x \, dy \) is independent of the path.

(b) Evaluate the integral in part (a) along the line segment from \((0, 1)\) to \((\pi, -1)\).

(c) Evaluate the integral \( \int_{(0,1)}^{(\pi,-1)} y \sin x \, dx - \cos x \, dy \) using Theorem 16.1, and confirm that the value is the same as that obtained in part (b).

9 – 14 Show that the line integral is independent of the path, and use Theorem 16.1 to find its value.

9. \( \int_{(1,2)}^{(4,0)} 3y \, dx + 3x \, dy \)

10. \( \int_{(0,0)}^{(1,\pi/2)} e^x \sin y \, dx + e^x \cos y \, dy \)

11. \( \int_{(0,0)}^{(3,2)} 2xe^y \, dx + x^2 e^y \, dy \)

12. \( \int_{(2,-2)}^{(-1,0)} 2xy^3 \, dx + 3y^2 x^2 \, dy \)

13. \( \int_{(-1,2)}^{(0,1)} (3x - y + 1) \, dx - (x + 4y + 2) \, dy \)

14. \( \int_{(1,1)}^{(3,3)} \left( e^y \ln y - \frac{e^y}{x} \right) \, dx + \left( \frac{e^x}{y} - e^y \ln x \right) \, dy \), where \( x \) and \( y \) are positive.

15 – 18 Confirm that the force field \( \vec{F} \) is conservative in some open connected region containing the points \( P \) and \( Q \), and then find the work done by the force field on a particle moving along an arbitrary smooth curve in the region from \( P \) to \( Q \).

15. \( \vec{F}(x, y) = xy^2 \vec{i} + x^2 y \vec{j} ; P(1,1), \ Q(0,0) \)

16. \( \vec{F}(x, y) = 2xy^3 \vec{i} + 3x^2 y^2 \vec{j} ; P(-3,0), \ Q(4,1) \)

17. \( \vec{F}(x, y) = ye^{xy} \vec{i} + xe^{xy} \vec{j} ; P(-1,1), \ Q(2,0) \)

18. \( \vec{F}(x, y) = e^{-y} \cos x \vec{i} - e^{-y} \sin x \vec{j} ; P(\pi/2,1), \ Q(-\pi/2,0) \)

19 – 20 Find the exact value of \( \int_C \vec{F} \cdot d\vec{r} \) using any method.

19. \( \vec{F}(x, y) = (e^y + ye^x) \vec{i} + (xe^y + e^x) \vec{j} \)
   \( C : \vec{r}(t) = \sin(\pi t/2) \vec{i} + \ln t \vec{j} \) \( 1 \leq t \leq 2 \)

20. \( \vec{F}(x, y) = 2xy \vec{i} + (x^2 + \cos y) \vec{j} \)
   \( C : \vec{r}(t) = t \vec{i} + t \cos(t/3) \vec{j} \) \( 0 \leq t \leq \pi \)
Answers to Exercise 7.3

1. conservative \( \phi = \frac{x^2}{2} + \frac{y^2}{2} + K \)  
3. not conservative

6. conservative \( \phi = x \cos y + y \sin x + K \)  
9. \(-6\)  
11. \(9e^2\)  
12. 32  
15/ \( W = \frac{1}{2} \)

17. \(W = 1 - e^{-1}\)  
19. \(\ln 2 - 1\)

7.4  Green’s Theorem

In this section we will discuss a remarkable and beautiful theorem that expresses a double integral over a plane region in terms of a line integral around its boundary.

Green’s Theorem

**Theorem 7.5 (Green’s Theorem)**

Let \( R \) be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve \( C \) oriented counterclockwise. If \( f(x, y) \) and \( g(x, y) \) are continuous and have continuous first partial derivatives on some open set containing \( R \), then

\[
\oint_C f(x, y) \, dx + g(x, y) \, dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA \quad (7.27)
\]

**Example 7.16**

Use Green’s Theorem to evaluate \( \int_C x^2 y \, dx + x \, dy \) along the triangle path shown in Figure below.

Solution
A Notation for Line Integrals Around Simple Closed Curves

It is common practice to denote a line integral around a simple closed curve by an integral sign with a superimposed circle. With this notation Formula (7.27) would be written as

\[ \oint_C f(x, y) \, dx + g(x, y) \, dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA \]  
(7.28)

Finding Work Using Green’s Theorem

It follows from Formula (7.18) of Section 7.2 that the integral on the left side of (7.28) is the work performed by the force field

\[ \vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j} \]

on a particle moving counterclockwise around the simple closed curve \( C \). In the case where this vector field is conservative, it follows from Theorem 7.1 that the integrand in the double integral on the right side of (7.28) is zero, so the work performed by the field is zero, as expected. For vector fields that are not conservative, it is often more efficient to calculate the work around simple closed curve by using Green’s Theorem than by parametrizing the curve.

Example 7.17

Find the work done by the force field \( \vec{F}(x, y) = (e^x - y^3)\vec{i} + (\cos y + x^3)\vec{j} \) on a particle that travels once around the unit circle \( x^2 + y^2 = 1 \) in the counterclockwise direction.

Solution
Finding Areas Using Green’s Theorem

Green’s Theorem leads to some useful new formulas for the area $A$ of the region $R$ that satisfies the conditions of the theorem. Two such formulas can be obtained as follows:

$$A = \mathop{\int\int}_R dA = \oint_C x \, dy \quad \text{and} \quad A = \mathop{\int\int}_R dA = \oint_C (-y) \, dx$$

Set $f(x, y) = 0$ and $g(x, y) = x$ in (7.27)

Set $f(x, y) = -y$ and $g(x, y) = 0$ in (7.27)

A third formula can be obtained by adding these two equations together. Thus, we have the following three formulas that express the area $A$ of a region $R$ in terms of line integrals around the boundary:

$$A = \oint_C x \, dy = - \oint_C y \, dx = \frac{1}{2} \oint_C (-y) \, dx + x \, dy \quad (7.29)$$

Example 7.18

Use a line integral to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution
Exercise 7.4

1 – 2  Evaluate the line integral using Green’s Theorem and check the answer by evaluating it directly.

1. \( \oint_C y^2 \, dx + x^2 \, dy \), where \( C \) is the square with vertices \((0, 0), (1, 0), (1, 1), \) and \((0, 1)\) oriented counterclockwise.

2. \( \oint_C y \, dx + x \, dy \), where \( C \) is the unit circle oriented counterclockwise.

3 – 13  Use Green’s Theorem to evaluate the integral. In each exercise, assume that the curve \( C \) is oriented counterclockwise.

3. \( \oint_C 3xy \, dx + 2xy \, dy \), where \( C \) is the rectangle bounded by \( x = -2, \ x = 4, \ y = 1, \) and \( y = 2. \)

4. \( \oint_C (x^2 - y^2) \, dx + x \, dy \), where \( C \) is the circle \( x^2 + y^2 = 9. \)

5. \( \oint_C x \cos y \, dx - y \sin x \, dy \), where \( C \) is the square with vertices \((0, 0), (\pi/2, 0), (\pi/2, \pi/2), \) and \((0, \pi/2)\).

6. \( \oint_C y \tan^2 x \, dx + \tan x \, dy \), where \( C \) is the circle \( x^2 + (y + 1)^2 = 1. \)

7. \( \oint_C (x^2 - y) \, dx + x \, dy \), where \( C \) is the circle \( x^2 + y^2 = 4. \)

8. \( \oint_C (e^x + y^2) \, dx + (e^y + x^2) \, dy \), where \( C \) is the boundary of the region between \( y = x^2 \) and \( y = x. \)

9. \( \oint_C \ln(1 + y) \, dx - \frac{xy}{1+y} \, dy \), where \( C \) is the triangle with vertices \((0, 0), (2, 0), \) and \((0, 4)\).

10. \( \oint_C x^2 \, dy - y^2 \, dx \), where \( C \) is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle \( x^2 + y^2 = 16. \)
11. \( \oint_C \tan^{-1} y \, dx - \frac{y^2 x}{1 + y^2} \, dy \), where \( C \) is the square with vertices \((0,0), (1,0), (1,1), \) and \((0,1)\).

12. \( \oint_C \cos x \sin y \, dx + \sin x \cos y \, dy \), where \( C \) is the triangle with vertices \((0,0), (3,3), \) and \((0,3)\).

13. \( \oint_C x^2 y \, dx + (y + xy^2) \, dy \), where \( C \) is the boundary of the region enclosed by \( y = x^2 \) and \( x = y^2 \).

14. Use a line integral to find the area enclosed by the astroid
\[
 x = a \cos^3 \phi, \quad y = a \sin^3 \phi \quad (0 \leq \phi \leq 2\pi)
\]

15. Use a line integral to find the area of the triangle with vertices \((0,0), (a,0), \) and \((0,b)\), where \( a > 0 \) and \( b > 0 \).

16 – 17 Use Green’s Theorem to find the work done by the force field \( \vec{F} \) on a particle that moves along the stated path.

16. \( \vec{F}(x,y) = xy \vec{i} + \left( \frac{1}{2} x^2 + xy \right) \vec{j} \); the particle starts at \((5,0)\), traverses the upper semicircle \( x^2 + y^2 = 25 \), and returns to its starting point along the \( x \)-axis.

17. \( \vec{F}(x,y) = \sqrt{y} \vec{i} + \sqrt{x} \vec{j} \); the particle moves counterclockwise one time around the closed curve given by the equations \( y = 0, \ x = 2, \) and \( y = x^3/4 \).

**Answers to Exercise 7.4**

1. 0 3. 0 5. 0 7. 8\pi 9. -4 11. -1 13. 0 16. \( \frac{250}{3} \)

### 7.5 Surface Integrals

**Definition of a Surface Integral**

In this section we will define what is means to integrate a function \( f(x, y, z) \) over a smooth parametric surface \( \sigma \). We will consider the problem of finding the mass of a curved lamina whose density function (mass per unit area) is known. Recall that we defined a lamina to be an idealized flat object that is thin enough to be viewed as a plane region.

If \( \sigma \) models a lamina and if \( f(x, y, z) \) is the density function of the lamina, then the mass \( M \) of the lamina is given by

\[
 M = \iint_{\sigma} f(x, y, z) \, dS \quad (7.30)
\]
That is, to obtain the mass of a lamina, we integrate the density function over the smooth surface that models the lamina.

Note that if \( \sigma \) is a smooth surface of surface area \( S \), and \( f \) is identically 1, then

\[
\iint_{\sigma} dS = S
\]  

(7.31)

Evaluating Surface Integrals

The following theorem provides a method for evaluating a surface integral when \( \sigma \) is represented parametrically.

**Theorem 7.6**

Let \( \sigma \) be a smooth parametric surface whose vector equation is

\[
\vec{r} = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}
\]

where \((u, v)\) varies over a region \( R \) in the \( uv \)-plane. If \( f(x, y, z) \) is continuous on \( \sigma \), then

\[
\iint_{\sigma} f(x, y, z) dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA
\]  

(7.32)

**Surface Integrals over** \( z = g(x, y) \), \( y = g(x, z) \), and \( x = g(y, z) \)

In the case where \( \sigma \) is a surface of the form \( z = g(x, y) \), we can take \( x = u \) and \( y = v \) as parameters and express the equation of the surface as

\[
\vec{r} = u\vec{i} + v\vec{j} + g(u, v)\vec{k}
\]

in which case we obtain

\[
\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1}
\]

Thus, it follows from (7.32) that

\[
\iint_{\sigma} f(x, y, z) dS = \iint_{R} f(x, y, g(x, y)) \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} dA
\]

Note that in this formula the region \( R \) lies in the \( xy \)-plane because the parameters are \( x \) and \( y \). Geometrically, this region is the projection of \( \sigma \) on the \( xy \)-plane. Here is a sketch of some surface \( \sigma \).
The following theorem summarizes this result and gives analogous formulas for surface integrals over surfaces of the form $y = g(x, z)$ and $x = g(y, z)$.

**Theorem 7.7**

(a) Let $\sigma$ be a surface with equation $z = g(x, y)$ and let $R$ be its projection on the $xy$-plane. If $g$ has continuous first partial derivatives on $R$ and $f(x, y, z)$ is continuous on $\sigma$, then

$$\iint_{\sigma} f(x, y, z) \, dS = \iint_{R} f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$  \hspace{1cm} (7.33)

(b) Let $\sigma$ be a surface with equation $y = g(x, z)$ and let $R$ be its projection on the $xz$-plane. If $g$ has continuous first partial derivatives on $R$ and $f(x, y, z)$ is continuous on $\sigma$, then

$$\iint_{\sigma} f(x, y, z) \, dS = \iint_{R} f(x, g(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1} \, dA$$  \hspace{1cm} (7.34)

(c) Let $\sigma$ be a surface with equation $x = g(y, z)$ and let $R$ be its projection on the $yz$-plane. If $g$ has continuous first partial derivatives on $R$ and $f(x, y, z)$ is continuous on $\sigma$, then

$$\iint_{\sigma} f(x, y, z) \, dS = \iint_{R} f(g(y, z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1} \, dA$$  \hspace{1cm} (7.35)
Example 7.19

Evaluate the surface integral \( \iint_{\sigma} xz \, dS \) where \( \sigma \) is the part of the plane \( x + y + z = 1 \) that lies in the first octant.

Solution
Evaluate the surface integral \( \iint_{\sigma} y^2 z^2 \, dS \) where \( \sigma \) is the part of the cone 
\[ z = \sqrt{x^2 + y^2} \] 
that lies between the planes \( z = 1 \) and \( z = 2 \).

**Example 7.20**

**Solution**

Suppose that a curved lamina \( \sigma \) with constant density \( \delta(x, y, z) = \delta_0 \) is the portion of the paraboloid \( z = x^2 + y^2 \) below the plane \( z = 1 \). Find the mass of the lamina.

**Example 7.21**

**Solution**
Exercise 7.5

1 - 3 Evaluate the surface integral \( \iint_{\sigma} f(x, y, z) \, dS \)

1. \( f(x, y, z) = z^2 \); \( \sigma \) is a portion of the cone \( z = \sqrt{x^2 + y^2} \) between the planes \( z = 1 \) and \( z = 3 \).

2. \( f(x, y, z) = x + y \); \( \sigma \) is the first-octant portion of the plane \( 2x + 3y + z = 6 \).

3. \( f(x, y, z) = x - y - z \); \( \sigma \) is a portion of the plane \( x + y = 1 \) in the first octant between \( z = 0 \) and \( z = 1 \).

4 - 5 Set up, but do not evaluate, an iterated integral equal to the given surface integral by projecting \( \sigma \) on (a) the \( yz \)-plane and (b) the \( xz \)-plane.

4. \( \iint_{\sigma} xyz \, dS \) where \( \sigma \) is a portion of the plane \( 2x + 3y + 4z = 12 \) in the first octant.

5. \( \iint_{\sigma} (x^2 - 2y + z) \, dS \) where \( \sigma \) is the portion of the graph of \( 4x + y = 8 \) bounded by the coordinate planes and the plane \( z = 6 \).

6 - 7 Find the mass of the lamina with constant density \( \delta_0 \).

6. The lamina that is the portion of the circular cylinder \( x^2 + z^2 = 4 \) that lies directly above the rectangle \( R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 4\} \) in the \( xy \)-plane.

7. The lamina that is the portion of the paraboloid \( 2z = x^2 + y^2 \) inside the cylinder \( x^2 + y^2 = 8 \).

8. Find the mass of the lamina that is the portion of the surface \( y^2 = 4 - z \) between the planes \( x = 0 \), \( x = 3 \), \( y = 0 \), and \( y = 3 \) if the density is \( \delta(x, y, z) = y \).

Answers to Exercise 7.5

1. \( \frac{15}{2} \pi \sqrt{2} \) 2. \( 5\sqrt{14} \) 3. \( -\frac{\sqrt{2}}{2} \)

4. (a) \( \frac{\sqrt{29}}{4} \int_{0}^{3} \int_{0}^{(12-4z)/3} yz(12 - 3y - 4z) \, dy \, dz \) \hspace{1cm} (b) \( \frac{29}{9} \int_{0}^{3} \int_{0}^{6-2z} xz(12-2x-4z) \, dx \, dz \)

5. (a) \( \frac{\sqrt{17}}{4} \int_{0}^{8} \int_{0}^{6} \left( 4 - 3y + \frac{y^2}{16} + z \right) \, dy \, dz \) \hspace{1cm} (b) \( \sqrt{17} \int_{0}^{2} \int_{0}^{6} \left[ x^2 - 2(8-4x) + z \right] \, dx \, dz \)

6. \( \frac{4}{3} \pi \delta_0 \) 8. \( \frac{1}{4}(37\sqrt{37} - 1) \)
The Divergence Theorem

In Section 7.1 we defined the divergence of a vector field

\[ \vec{F}(x, y, z) = f(x, y, z)\vec{i} + g(x, y, z)\vec{j} + h(x, y, z)\vec{k} \]

as

\[ \text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \]

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as the Divergence Theorem or Gauss's Theorem, will provide us with a physical interpretation of divergence in the context of fluid flow.

**Theorem 7.8 (The Divergence Theorem)**

Let \( G \) be a solid whose surface \( \sigma \) is oriented outward. If

\[ \vec{F}(x, y, z) = f(x, y, z)\vec{i} + g(x, y, z)\vec{j} + h(x, y, z)\vec{k} \]

where \( f, g, \) and \( h \) have continuous first partial derivatives on some open set containing \( G \), and if \( \vec{n} \) is the outward unit normal on \( \sigma \), then

\[ \iint_{\sigma} \vec{F} \cdot \vec{n} \, dS = \iiint_{G} \text{div } \vec{F} \, dV \]

The flux of a vector field across a closed surface with outward orientation is sometimes called the outward flux across the surface. In word, the Divergence Theorem states:

**The outward flux of a vector field across a closed surface is equal to the triple integral of the divergence over the region enclosed by the surface.**

Using the Divergence Theorem to Find Flux

**Example 7.22**

Use the Divergence Theorem to find the outward flux of the vector field \( \vec{F}(x, y, z) = z\vec{k} \) across the sphere \( x^2 + y^2 + z^2 = a^2 \).

Solution
Example 7.23

Use the Divergence Theorem to find the outward flux of the vector field

$$\vec{F}(x, y, z) = 2x\vec{i} + 3y\vec{j} + z^2\vec{k}$$

across the unit cube defined by the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

Example 7.24

Use the Divergence Theorem to find the outward flux of the vector field

$$\vec{F}(x, y, z) = x^3\vec{i} + y^3\vec{j} + z^2\vec{k}$$

across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the plane $z = 0$ and $z = 2$.

Solution
Exercise 7.6

1 − 4 Use the Divergence Theorem to find \( \int \int_{\sigma} \mathbf{F} \cdot \hat{n} \, dS \)

1. \( \mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \); \( \sigma \) is the surface of the cube bounded by the plane \( x = 0, x = 1, y = 0, y = 1, z = 0, z = 1 \).

2. \( \mathbf{F}(x, y, z) = y \sin x \mathbf{i} + y^2 z \mathbf{j} + (x + 3z) \mathbf{k} \); \( \sigma \) is a surface of the region bounded by the plane \( x = \pm 1, y = \pm 1, z = \pm 1 \).

3. \( \mathbf{F}(x, y, z) = (x^2 + \sin yz) \mathbf{i} + (y - xe^{-z}) \mathbf{j} + z^2 \mathbf{k} \); \( \sigma \) is a surface of the region bounded by the cylinder \( x^2 + y^2 = 4 \) and the planes \( x + z = 2 \) and \( z = 0 \).

4. \( \mathbf{F}(x, y, z) = 2x \mathbf{i} - yz \mathbf{j} + z^2 \mathbf{k} \); the surface \( \sigma \) is the paraboloid \( z = x^2 + y^2 \) capped by the disk \( x^2 + y^2 \leq 1 \) in the plane \( z = 1 \).

5 − 10 Use the Divergence Theorem to find the flux of \( \mathbf{F} \) across the surface \( \sigma \) with outward orientation.

5. \( \mathbf{F}(x, y, z) = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \); \( \sigma \) is the graph of \( x^{2/3} + y^{2/3} + z^{2/3} = 1 \).

6. \( \mathbf{F}(x, y, z) = (x^2 + y) \mathbf{i} + z^2 \mathbf{j} + (e^y - z) \mathbf{k} \); \( \sigma \) is a surface of the rectangular solid bounded by the coordinate planes and the plane \( x = 3, y = 1, \) and \( z = 2 \).

7. \( \mathbf{F}(x, y, z) = (x - z) \mathbf{i} + (y - x) \mathbf{j} + (z - y) \mathbf{k} \); \( \sigma \) is a surface of the cylindrical solid bounded by \( x^2 + y^2 = a^2, z = 0, \) and \( z = 1 \).

8. \( \mathbf{F}(x, y, z) = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k} \); \( \sigma \) is a surface of the cylindrical solid bounded by \( x^2 + y^2 = 4, z = 0, \) and \( z = 3 \).

9. \( \mathbf{F}(x, y, z) = 3x \mathbf{i} + xz \mathbf{j} + z^2 \mathbf{k} \); \( \sigma \) is the surface of the region bounded by the paraboloid \( z = 4 - x^2 - y^2 \) and the \( xy \)-plane.

10. \( \mathbf{F}(x, y, z) = 2xz \mathbf{i} + xyz \mathbf{j} + yz \mathbf{k} \); \( \sigma \) is a surface of the region bounded by the coordinate planes and the planes \( x + 2z = 4 \) and \( y = 2 \).

Answers to Exercise 7.6

1. 3 2. 24 3. 20\pi 4. \frac{4\pi}{3} 5. 0 6. 12 7. 3\pi a^2 8. 180\pi 9. \frac{136\pi}{3} 10. 24

7.7 Stokes’ Theorem

In Section 7.1 we defined the curl of a vector field

\( \mathbf{F}(x, y, z) = f(x, y, z) \mathbf{i} + g(x, y, z) \mathbf{j} + h(x, y, z) \mathbf{k} \)
As
\[
\text{curl } \vec{F} = \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}
\]

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as **Stokes’ Theorem**, will provide us with a physical interpretation of curl in the context of fluid flow.

**Theorem 7.9 (Stokes’ Theorem)**

Let \( \sigma \) be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve \( C \) with positive orientation. If the components of the vector field
\[
\vec{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}
\]
are continuous and have continuous first partial derivatives on some open set containing \( \sigma \), and if \( \vec{T} \) is the unit tangent vector to \( C \), then
\[
\oint_C \vec{F} \cdot \vec{T} \, ds = \iint_{\sigma} (\text{curl } \vec{F}) \cdot \vec{n} \, dS
\]

If \( \sigma \) is a surface given by \( z = f(x, y) \) and oriented up, then
\[
\iint_{\sigma} (\text{curl } \vec{F}) \cdot \vec{n} \, dS = \iint_{R} (\text{curl } \vec{F}) \cdot \left( -\frac{\partial z}{\partial x} \hat{i} - \frac{\partial z}{\partial y} \hat{j} + \hat{k} \right) \, dA,
\]
where \( R \) is the projection of \( \sigma \) on the \( xy \)-plane.

If \( \sigma \) is a surface given by \( z = f(x, y) \) and oriented down, then
\[
\iint_{\sigma} (\text{curl } \vec{F}) \cdot \vec{n} \, dS = \iint_{R} (\text{curl } \vec{F}) \cdot \left( \frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} - \hat{k} \right) \, dA,
\]
where \( R \) is the projection of \( \sigma \) on the \( xy \)-plane.

Recall that if \( \vec{F} \) is a force field, the integral on the left side represents the work performed by the force field on a particle that traverses the curve \( C \). Thus Stokes’ Theorem states:

*The work performed by the force field on a particle that traverses a simple, closed, piecewise smooth curve \( C \) in the positive direction can be obtained by integrating the normal component of the curl over an oriented surface \( \sigma \) bounded by \( C \).*
Using Stokes’ Theorem to Calculate Work

For computational purposes it is usually preferable to use Formula (7.19) in Section 7.2 to rewrite the formula in Stoke’s Theorem as

\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_{\sigma} (\text{curl} \vec{F}) \cdot \vec{n} \, dS \]

Stokes’ Theorem is usually the method of choice for calculating work around piecewise smooth curves with multiple sections, since it eliminates the need for a separate integral evaluation over each section. This is illustrated in the following example.

**Example 7.25**

Find the work performed by the force field \( \vec{F}(x, y, z) = x^2 \vec{i} + 4xy^3 \vec{j} + xy^2 \vec{k} \) on a particle that traverses the rectangle \( C \) in the plane \( z = y \) shown in Figure below.

![Diagram of a rectangular path in a plane z = y]

**Solution**
Example 7.26

Use Stokes’ Theorem to evaluate \( \int_{\sigma} (\text{curl } \vec{F}) \cdot \vec{n} \, dS \) where

\[
\vec{F}(x, y, z) = 2z\vec{i} + 3x\vec{j} + 5y\vec{k},
\]

\( \sigma \) is the portion of the paraboloid \( z = 4 - x^2 - y^2 \) for which \( z \geq 0 \) with upward orientation, and \( C \) is the positively oriented circle \( x^2 + y^2 = 4 \) that form the boundary of \( \sigma \) in the \( xy \)-plane.

Solution
Exercise 7.7

1 – 3 Verify the Stokes’ Theorem by evaluating the line integral and the double integral. Assume that the surface has an upward orientation.

1. \( \vec{F}(x, y, z) = (x-y) \vec{i} + (y-z) \vec{j} + (z-x) \vec{k} \); \( \sigma \) is the portion of the plane \( x+y+z = 1 \) in the first octant.

2. \( \vec{F}(x, y, z) = y^2 \vec{i} + z^2 \vec{j} + x^2 \vec{k} \); \( \sigma \) is the first-octant portion of the plane \( x+y+z = 1 \).

3. \( \vec{F}(x, y, z) = z \vec{i} + x \vec{j} + y \vec{k} \); \( \sigma \) is the upper hemisphere \( z = \sqrt{a^2 - x^2 - y^2} \).

4 – 7 Use the Stokes’ Theorem to evaluate \( \oint_C \vec{F} \cdot d\vec{r} \).

4. \( \vec{F}(x, y, z) = z^2 \vec{i} + 2x \vec{j} - y^3 \vec{k} \); \( C \) is the circle \( x^2 + y^2 = 1 \) in the \( xy \)-plane with counterclockwise orientation looking down the positive \( z \)-axis.

5. \( \vec{F}(x, y, z) = 3z \vec{i} + 4x \vec{j} + 2y \vec{k} \); \( C \) is the boundary of the paraboloid \( z = 4 - x^2 - y^2 \) for which \( z \geq 0 \) with counterclockwise orientation looking down the positive \( z \)-axis.

6. \( \vec{F}(x, y, z) = xy \vec{i} + x^2 \vec{j} + z^2 \vec{k} \); \( C \) is the intersection of the paraboloid \( z = x^2 + y^2 \) and the plane \( z = y \) with a counterclockwise orientation looking down the positive \( z \)-axis.

7. \( \vec{F}(x, y, z) = (x-y) \vec{i} + (y-z) \vec{j} + (z-x) \vec{k} \); \( C \) is the circle \( x^2 + y^2 = a^2 \) in the \( xy \)-plane with counterclockwise orientation looking down the positive \( z \)-axis.

8. If \( \vec{F}(x, y, z) = 2y \vec{i} + e^z \vec{j} - \tan^{-1} x \vec{k} \) and \( \sigma \) is the portion of the paraboloid \( z = 4 - x^2 - y^2 \) cut off by the \( xy \)-plane, use Stokes’ Theorem to evaluate \( \iint_{\sigma} (\text{curl} \vec{F}) \cdot \vec{n} \, dS \).

Answers to Exercise 7.7

1. \( \frac{3}{2} \) 2. \( -1 \) 3. \( \pi a^2 \) 4. \( 2\pi \) 5. \( 16\pi \) 6. 0 7. \( \pi a^2 \) 8. \( -8\pi \)

END