Chapter 2

Parametric and Polar Curves

2.1 Parametric Equations; Tangent Lines and Arc Length for Parametric Curves

Parametric Equations

So far we’ve described a curve by giving an equation that the coordinates of all points on the curve must satisfy. For example, we know that the equation \( y = x^2 \) represents a parabola in rectangular coordinates. We now study another method for describing a curve in the plane. In this method, the \( x \)- and \( y \)-coordinates of points on the curve are given separately as functions of an additional variable \( t \), called the parameter:

\[
x = f(t), \quad y = g(t)
\]

These are called parametric equations for the curve.

Each value of \( t \) determines a point \((x, y)\), which we can plot in a coordinate plane. As \( t \) varies, the point \((x, y) = (f(t), g(t))\) varies and traces out a curve \( C \), which we call a parametric curve.

Example 2.1

Sketch the curve defined by the parametric equations

\[
x = t^2 - 3t, \quad y = t - 1
\]
Solution For every value of $t$, we get a point on the curve. Here, we plot the points $(x, y)$ determined by the values of $t$ shown in the following table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>10</td>
<td>-3</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>4</td>
</tr>
</tbody>
</table>

Notice that as $t$ increases a particle whose position is given by the parametric equations moves along the curve in the direction of the arrows. Thus, a parametrization contain more information than just the curve being parametrized; it also indicates how the curve is being traced out.

Example 2.2

Describe and graph the curve represented by the parametric equations

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi) \quad (2.1)$$

Solution One way to identify the curve, we eliminate the parameter. Since $\cos^2 t + \sin^2 t = 1$ and since $x = \cos t$ and $y = \sin t$ for every point $(x, y)$ on the curve, we have

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

This means that all points on the curve satisfy the equation $x^2 + y^2 = 1$, so the graph is a circle of radius 1 centered at the origin. As $t$ increases from 0 to $2\pi$, the point given by the parametric equations starts at $(1, 0)$ and moves counterclockwise once around the circle, as shown in Figure below. Notice that the parameter $t$ can be interpreted as the angle shown in the figure.
Orientation

The direction in which the graph of a pair of parametric equations is traces as the parameter increases is called the direction of increasing parameter or sometimes the orientation imposed on the curve by the equation. Thus, we make a distinction between a curve, which is the set of points, and a parametric curve, which is a curve with an orientation.

For example, we saw in Example 2.2 that the circle represented parametrically by (2.1) is traced counterclockwise as $t$ increases and hence has counterclockwise orientation.

Example 2.3

Sketch and identify the curve defined by the parametric equations

$$x = \sin^2 t, \quad y = 2 \cos t$$

by eliminating the parameter, and indicate the orientation on the graph.

Solution

Expressing Ordinary Functions Parametrically

An equation $y = f(x)$ can be expressed in parametric form by introducing the parameter $t = x$; this yields the parametric equations

$$x = t, \quad y = f(t).$$

For example, the curve $y = \cos x$ over the interval $[-2\pi, 2\pi]$ can be expressed parametrically as

$$x = t, \quad y = \cos t \quad (-2\pi \leq t \leq 2\pi).$$

If a function $f$ is one-to-one, then it has an inverse function $f^{-1}$. In this case the equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. We can express the graph of $f^{-1}$ in parametric form by introducing the parameter $t = y$; this yields the parametric equations

$$x = f(t), \quad y = t.$$
For example, the graph of \( f(x) = x^5 + x + 1 \) can be represented parametrically as

\[ x = t, \quad y = t^5 + t + 1 \]

and the graph of \( f^{-1} \) can be represented parametrically as

\[ x = t^5 + t + 1, \quad y = t. \]

### Tangent Line to Parametric Curves

When a curve is described by an equation of the form \( y = f(x) \), we know that the slope of the tangent line of the curve at the point \((x_0, y_0)\) is given by

\[
\frac{dy}{dx} \bigg|_{x=x_0} = f'(x_0).
\]

However, if the curve is defined by parametric equations

\[ x = f(t), \quad y = g(t), \]

then we may not have a description of the curve as a function of \( x \) in order to compute the slope of the tangent line in this way. Instead, we apply the Chain Rule to obtain

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}
\]

Solving for \( \frac{dy}{dx} \) yields

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{2.2}
\]

This allows us to express \( \frac{dy}{dx} \) as a function of the parameter \( t \).

---

**Example 2.4**

Find the slope of the tangent line to the curve

\[ x = t - 3 \sin t, \quad y = 4 - 3 \cos t \quad (t \geq 0) \]

at the point where \( t = \pi/3 \).

**Solution**
We know that a curve defined by the equation $y = f(x)$ has a horizontal tangent if $dy/dx = 0$, and a vertical tangent if $f'(x)$ has a vertical asymptote.

For parametric curves, we also can identify a **horizontal tangent** by determining where $dy/dx = 0$. This is the case whenever $dy/dt = 0$, provided that $dx/dt \neq 0$. Similarly, the curve has infinite slope and a **vertical tangent line** whenever $dx/dt = 0$, but $dy/dt \neq 0$. At points where $dx/dt$ and $dy/dt$ are both zero, $dy/dx$ is the indeterminate form $0/0$; we call such points **singular points**.

**Example 2.5**

A curve $C$ is defined by the parametric equations

$$x = t^2 \quad \text{and} \quad y = t^3 - 3t.$$

Find the points on $C$ where the tangent is horizontal or vertical.

**Solution**

**Example 2.6**

The curve represented by the parametric equations

$$x = t^2 \quad y = t^3 \quad (-\infty < t < +\infty)$$

is called a **semicubical parabola**. The parameter $t$ can be eliminated by cubing $x$ and squaring $y$, from which it follows that $y^2 = x^3$.

The graph of this equation consists of two branches: an upper branch obtained by graphing $y = x^{3/2}$ and a lower branch obtained by graphing $y = -x^{3/2}$. The two branches meet at the origin, which corresponds to $t = 0$ in the parametric equations. This is a singular point because the derivatives $dx/dt = 2t$ and $dy/dt = 3t^2$ are both zero there.
Without eliminating the parameter, find $dy/dx$ and $d^2y/dx^2$ at $(1, -1)$ on the semicubical parabola given by the parametric equations in Example 2.6

Solution

Arc Length of Parametric Curve

The following result provides a formula for finding the arc length of a curve from parametric equations for the curve.

Arc length formula for parametric curve

If no segment of the curve represented by the parametric equations

\[ x = x(t), \quad y = y(t) \quad (a \leq t \leq b) \]

is traced more than once as $t$ increase from $a$ to $b$, and if $dx/dt$ and $dy/dt$ are continuous functions for $a \leq t \leq b$, then the arc length $L$ of the curve is given by

\[ L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \quad (2.3) \]

Example 2.8

Find the circumference of a circle of radius $a$ from the parametric equations

\[ x = a \cos t, \quad y = a \sin t \quad (0 \leq t \leq 2\pi) \]

Solution
Exercise 2.1

1. Find the slope of the tangent line to the parametric curve \( x = \frac{t}{2}, \ y = t^2 + 1 \) at \( t = -1 \) and at \( t = 1 \) without eliminating the parameter.

2. Find the slope of the tangent line to the parametric curve \( x = 3 \cos t, \ y = 4 \sin t \) at \( t = \pi/4 \) and at \( t = 7\pi/4 \) without eliminating the parameter.

3. Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) at the given point without eliminating the parameter.

3 - 8  \[ x = \sqrt{t}, \ y = 2t + 4; \ t = 1 \]

4. \[ x = \frac{1}{2} t^2 + 1, \ y = \frac{1}{3} t^3 - t; \ t = 2 \]

5. \[ x = \sec t, \ y = \tan t; \ t = \pi/3 \]

6. \[ x = \sinh t, \ y = \cosh t; \ t = 0 \]

7. \[ x = \theta + \cos \theta, \ y = 1 + \sin \theta; \ \theta = \pi/6 \]

8. \[ x = \cos \phi, \ y = 3 \sin \phi; \ \phi = 5\pi/6 \]

9. (a) Find the equation of the tangent line to the curve \[ x = e^t, \ y = e^{-t} \] at \( t = 1 \) without eliminating the parameter.

(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

10. (a) Find the equation of the tangent line to the curve \[ x = 2t + 4, \ y = 8t^2 - 2t + 4 \] at \( t = 1 \) without eliminating the parameter.

(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

11 - 12  Find all values of \( t \) at which the parametric curve has (a) a horizontal tangent line and (b) a vertical tangent line.

11. \[ x = 2 \sin t, \ y = 4 \cos t \quad (0 \leq t \leq 2\pi) \]

12. \[ x = 2t^3 - 15t^2 + 24t + 7, \ y = t^2 + t + 1 \]

Answers to Exercise 2.1

1. \(-4, 4 \quad 3. 4, 4 \quad 5. 2/\sqrt{3}, -1/ (3\sqrt{3}) \quad 7. \sqrt{3}, 4 \quad 9. y = -e^{-2x} + 2e^{-1} \)

11. (a) 0, \pi, 2\pi  \quad (b) \pi/2, 3\pi/2
### 2.2 Polar Coordinates

**Definition 2.1**

The **polar coordinate system** in a plane consists of a fixed point \( O \), called the **pole** (or **origin**), and a ray emanating from a pole, called the **polar axis**. Each point \( P \) on a plane is determined by a distance \( r \) from \( P \) to the pole and an angle \( \theta \) from the polar axis to the ray \( OP \). The point \( P \) is represented by the ordered pair \((r, \theta)\) and \( r, \theta \) are called **polar coordinates**.

![Polar Coordinate System](image)

**Remark**: We extend the meaning of polar coordinates \((r, \theta)\) to the case in which \( r \) is negative by agreeing that the points \((-r, \theta)\) and \((r, \theta)\) lie in the same line through \( O \) and at the same distance \(|r|\) from \( O \), but on opposite sides of \( O \). If \( r > 0 \), the point \((r, \theta)\) lies in the same quadrant as \( \theta \); if \( r < 0 \), it lies in the quadrant on the opposite side of the pole.

![Extended Polar Coordinates](image)

**Example 2.9**

Plot the points whose polar coordinates are given.

(a) \((3, \pi/4)\) 
(b) \((4, 2\pi/3)\)
(c) \((2, 5\pi/4)\) 
(d) \((4, 11\pi/6)\)

**Solution**
Remark: In the rectangular coordinate system (or Cartesian coordinate system) every point has only one representation, but in the polar coordinate system each point has many representations. For example, the polar coordinates

$$(1, 5\pi/3), \ (1, -\pi/3), \ (1, 11\pi/3)$$

all represent the same point.

In general, if a point $P$ has polar coordinates $(r, \theta)$, then

$$(r, \theta + 2n\pi), \ n = 0, \pm 1, \pm 2, \ldots$$

and

$$(-r, \theta + (2n-1)\pi), \ n = 0, \pm 1, \pm 2, \ldots$$

are also polar coordinates of $P$.

**Example 2.10**

Find all the polar coordinates of the point $P(3, \pi/6)$.

Solution

**Converting from Polar to Rectangular Form, and Vice Versa**

Often, it is necessary to transform coordinates or equations in rectangular form to polar form, or vice versa. The connection between polar and rectangular coordinates can be seen from the Figure below and described by the following formulas:

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}$$

**Note:** The signs of $x$ and $y$ determine the quadrant for $\theta$. 
Example 2.11

(a) Find the rectangular coordinates of the point $P$ whose polar coordinates are $(4, 2\pi/3)$.

(b) Find polar coordinates of the point $P$ whose rectangular coordinates are $(1, -1)$.

Solution

Example 2.12

(a) Express the equation $x^2 = 4y$ in polar coordinates.

(b) Change $r = -3\cos\theta$ to rectangular form.

Solution
Polar Curves

We now turn to graphing polar equations. The graph of a polar equation, such as \( r = 3\theta \) or \( r = 6\cos\theta \), in a polar coordinate system is the set of all points having coordinates that satisfy the polar equation.

**Example 2.13**

Sketch the graph of (a) \( r = 1 \) and (b) \( \theta = \pi/4 \)

**Solution**

**Example 2.14**

Sketch the graph of \( r = 2\cos\theta \) in polar coordinates by plotting points.

**Solution** We construct a table using some convenient values of \( \theta \), plot these points, then join the points with a smooth curve:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r = 2\cos\theta )</th>
<th>( (r, \theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>( \pi/6 )</td>
<td>( \sqrt{3} )</td>
<td>(( \sqrt{3}, \pi/2 ))</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>1</td>
<td>(1, ( \pi/3 ))</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>0</td>
<td>(0, ( \pi/2 ))</td>
</tr>
<tr>
<td>( 2\pi/3 )</td>
<td>-1</td>
<td>(-1, ( 2\pi/3 ))</td>
</tr>
<tr>
<td>( 5\pi/6 )</td>
<td>-( \sqrt{3} )</td>
<td>(-( \sqrt{3}, 5\pi/6 ))</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-2</td>
<td>(-2, ( \pi ))</td>
</tr>
</tbody>
</table>
Sketch the graph of $r = \cos 2\theta$ (four-petal rose) in polar coordinates.

Solution

\[
\begin{align*}
0 \leq \theta & \leq \frac{\pi}{6} \\
0 \leq \theta & \leq \frac{2\pi}{6} \\
0 \leq \theta & \leq \frac{3\pi}{6} \\
0 \leq \theta & \leq \frac{4\pi}{6} \\
0 \leq \theta & \leq \frac{5\pi}{6} \\
0 \leq \theta & \leq \frac{6\pi}{6} \\
0 \leq \theta & \leq \frac{7\pi}{6} \\
0 \leq \theta & \leq \frac{8\pi}{6} \\
0 \leq \theta & \leq \frac{9\pi}{6} \\
0 \leq \theta & \leq \frac{10\pi}{6} \\
0 \leq \theta & \leq \frac{11\pi}{6} \\
0 \leq \theta & \leq \frac{12\pi}{6}
\end{align*}
\]
Match the polar equations with the graphs:

- $r = \sin(\theta/2)$
- $r = \sin(\theta/4)$
- $r = 1 + 5\cos 5\theta$

### Symmetry Tests

Observe that the polar graph $r = \cos 2\theta$ in Example 2.15 is symmetric about the $x$-axis and the $y$-axis. This symmetry could have been predicted from the following theorem.

#### Theorem 2.1 (Symmetry Tests)

(a) A curve in polar coordinates is symmetric about the $x$-axis if replacing $\theta$ by $-\theta$ in its equation produces an equivalent equation.

(b) A curve in polar coordinates is symmetric about the $y$-axis if replacing $\theta$ by $\pi - \theta$ in its equation produces an equivalent equation.

(c) A curve in polar coordinates is symmetric about the origin if replacing $\theta$ by $\theta + \pi$, or replacing $r$ by $-r$ in its equation produces an equivalent equation.
Example 2.17

Show that the graph of \( r = \cos 2\theta \) is symmetric about the \( x \)-axis and \( y \)-axis.

Solution

Families of Lines and Rays Through the Pole

For any constant \( \theta_0 \), the equation

\[
\theta = \theta_0
\]

(2.4)

is satisfied by the coordinates of the form \( P(r, \theta_0) \), regardless of the value of \( r \). Thus, the equation represents the line that passes through the pole and makes an angle of \( \theta_0 \) with the polar axis.

Families of Circles

We will consider three families of circles in which \( a \) is assumed to be a positive constant:

\[
\begin{align*}
  r &= a \\
  r &= 2a \cos \theta \\
  r &= 2a \sin \theta
\end{align*}
\]

- The equation \( r = a \) represents a circle of radius \( a \), centered at the pole
- The equation \( r = 2a \cos \theta \) represents a circle of radius \( a \), centered on the \( x \)-axis and tangent to the \( y \)-axis at the origin.
- The equation $r = 2a \sin \theta$ represents a circle of radius $a$, centered on the $y$-axis and tangent to the $x$-axis at the origin.

Sketch the graphs of the following equations in polar coordinates.

(a) $r = 4 \cos \theta$  
(b) $r = -5 \sin \theta$  
(c) $r = 3$

Solution
Families of Rose Curves

In polar coordinates, equations of the form

\[ r = a \sin n\theta \quad \text{or} \quad r = a \cos n\theta \]

in which \( a > 0 \) and \( n \) is a positive integer represent families of flower-shaped curves called roses. The rose consists of \( n \) equally spaced petals of radius \( a \) if \( n \) is odd and \( 2n \) equally spaced petals of radius \( a \) if \( n \) is even.

\[ r = a \sin 2\theta \quad r = a \sin 3\theta \quad r = a \sin 4\theta \quad r = a \sin 5\theta \quad r = a \sin 6\theta \]

\[ r = a \cos 2\theta \quad r = a \cos 3\theta \quad r = a \cos 4\theta \quad r = a \cos 5\theta \quad r = a \cos 6\theta \]

Families of Cardioids and Limaçons

Equations with any of the four forms

\[ r = a \pm b \sin \theta \quad r = a \pm b \cos \theta \]

in which \( a > 0 \) and \( b > 0 \) represent polar curves called limaçons. There are four possible shapes for a limaçon that can be determined from the ratio \( a/b \). If \( a = b \) (the case \( a/b = 1 \)), then the limaçons is called a cardioids because of its heart-shaped appearance.

\[ a/b < 1 \quad a/b = 1 \quad 1 < a/b < 2 \quad a/b \geq 2 \]

Limaçon with inner loop  Cardioid  Dimpled limaçon  Convex limaçon

Example 2.19

Sketch the graph of the equation \( r = 2(1 - \cos \theta) \) in polar coordinates.

Solution
Families of Spirals

A spiral is a curve that coils around a central point. The most common example is the spiral of Archimedes, which has an equation of the form

\[ r = a \theta \quad (\theta \geq 0) \quad \text{or} \quad r = a \theta \quad (\theta \leq 0) \]

In these equations, \( \theta \) is in radians and \( a \) is positive.

Example 2.20

Sketch the graph of \( r = \theta \quad (\theta \geq 0) \) in polar coordinates by plotting points.

Solution

Exercise 2.2

1 – 2 Plot the points in polar coordinates.

1. (a) \((3, \pi/4)\)  
   (b) \((5, 2\pi/3)\)  
   (c) \((1, \pi/2)\)  
   (d) \((4, 7\pi/6)\)  
   (e) \((-6, -\pi)\)  
   (f) \((-1, 9\pi/4)\)

2. (a) \((2, -\pi/3)\)  
   (b) \((3/2, -7\pi/4)\)  
   (c) \((-3, 3\pi/2)\)  
   (d) \((-5, -\pi/6)\)  
   (e) \((2, 4\pi/3)\)  
   (f) \((0, \pi)\)

3 – 4 Find the rectangular coordinates of the points whose polar coordinates are given.

3. (a) \((6, \pi/6)\)  
   (b) \((7, 2\pi/3)\)  
   (c) \((-6, -5\pi/6)\)  
   (d) \((0, -\pi)\)  
   (e) \((7, 17\pi/6)\)  
   (f) \((-5, 0)\)

4. (a) \((-2, \pi/4)\)  
   (b) \((6, -\pi/4)\)  
   (c) \((4, 9\pi/4)\)  
   (d) \((3, 0)\)  
   (e) \((-4, -3\pi/2)\)  
   (f) \((0, 3\pi)\)

5. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \), and the second pair satisfying \( r \geq 0 \) and \(-2\pi < \theta \leq 0\).

   (a) \((-5, 0)\)  
   (b) \((2\sqrt{3}, -2)\)  
   (c) \((0, -2)\)  
   (d) \((-8, -8)\)  
   (e) \((-3, 3\sqrt{3})\)  
   (f) \((1, 1)\)
6. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are \((-\sqrt{3}, 1)\).

(a) \(r \geq 0\) and \(0 \leq \theta < 2\pi\)
(b) \(r \leq 0\) and \(0 \leq \theta < 2\pi\)
(c) \(r \geq 0\) and \(-2\pi < \theta \leq 0\)
(d) \(r \leq 0\) and \(-\pi < \theta \leq \pi\)

7. Identify the curve by transforming the given polar equation to rectangular coordinates.

7. (a) \(r = 2\)
(b) \(r \sin \theta = 4\)
(c) \(r = 3 \cos \theta\)
(d) \(r = \frac{6}{3 \cos \theta + 2 \sin \theta}\)

8. (a) \(r = 5 \sec \theta\)
(b) \(r = 2 \sin \theta\)
(c) \(r = 4 \cos \theta + 4 \sin \theta\)
(d) \(r = \sec \theta \tan \theta\)

9. Express the given equations in polar coordinates.

9. (a) \(x = 3\)
(b) \(x^2 + y^2 = 7\)
(c) \(x^2 + y^2 + 6y = 0\)
(d) \(9xy = 4\)

10. (a) \(y = -3\)
(b) \(x^2 + y^2 = 5\)
(c) \(x^2 + y^2 + 4x = 0\)
(d) \(x^2(x^2 + y^2) = y^2\)

11. Use the method of Example 2.14 to sketch the curve in polar coordinates.

11. \(r = 2(1 + \sin \theta)\)
12. \(r = 1 - \cos \theta\)

13. Sketch the curve in polar coordinates.

13. \(\theta = \frac{\pi}{3}\)
14. \(\theta = -\frac{3\pi}{4}\)
15. \(r = 3\)
16. \(r = 4 \cos \theta\)
17. \(r = 6 \sin \theta\)
18. \(r = 1 + \sin \theta\)
19. \(2r = \cos \theta\)
20. \(r - 2 = 2 \cos \theta\)
21. \(r = 3(1 + \sin \theta)\)
22. \(r = 5 - 5 \sin \theta\)
23. \(r = 4 - 4 \cos \theta\)
24. \(r = 1 + 2 \sin \theta\)
25. \(r = -1 - \cos \theta\)
26. \(r = 4 + 3 \cos \theta\)
27. \(r = 2 + \cos \theta\)
28. \(r = 3 - \sin \theta\)
29. \(r = 3 + 4 \cos \theta\)
30. \(r - 5 = 3 \sin \theta\)
31. \(r = 5 - 2 \cos \theta\)
32. \(r = -3 - 4 \sin \theta\)
33. \(r^2 = \cos 2\theta\)
34. \(r^2 = 9 \sin 2\theta\)
35. \(r^2 = 16 \sin 2\theta\)
36. \(r = 4\theta\) \((\theta \geq 0)\)
37. \(r = 4\theta\) \((\theta \leq 0)\)
38. \(r = 4\theta\)
39. \(r = -2 \cos 2\theta\)
40. \(r = 3 \sin 2\theta\)
41. \(r = 9 \sin 4\theta\)
42. \(r = 2 \cos 3\theta\)

43. Find the highest point on the cardioid \(r = 1 + \cos \theta\).
44. Find the leftmost point on the upper half of the cardioid \(r = 1 + \cos \theta\).
Answers to Exercise 2.2

3. (a) \((3\sqrt{3}, 3)\) (b) \((-7/2, 7\sqrt{3}/3)\) (c) \((3\sqrt{3}, 3)\) (d) \((0, 0)\) (e) \((-7\sqrt{3}/2, 7/2)\) (f) \((-5, 0)\)

5. (a) \((5, \pi), (5, -\pi)\) (b) \((4, 11\pi/6), (4, -\pi/6)\) (c) \((2, 3\pi/2), (2, -\pi/2)\) (d) \((8\sqrt{2}, 5\pi/4), (8\sqrt{2}, -3\pi/4)\) (e) \((6, 2\pi/3), (6, -4\pi/3)\) (f) \((\sqrt{2}, \pi/4), (\sqrt{2}, -7\pi/4)\)

7. (a) circle (b) line (c) circle (d) line

9. (a) \(r = 3 \sec \theta\) (b) \(r = \sqrt{7}\) (c) \(r = -6 \sin \theta\) (d) \(r^2 \cos \theta \sin \theta = 4/9\)

13. 

15. 

17. 

21. 

23. 

25. 

28. 

30. 

32.
2.3 Tangent Lines, Arc Length, and Area for Polar Curves

Tangents to Polar Curves

To find a tangent line to a polar curve \( r = f(\theta) \) we regard \( \theta \) as a parameter and write its parametric equations as

\[
\begin{align*}
    x &= r \cos \theta = f(\theta) \cos \theta \\
    y &= r \sin \theta = f(\theta) \sin \theta
\end{align*}
\]

Then, using the method for finding slopes of parametric curves and the Product Rule, we have

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} \tag{2.5}
\]

Example 2.21

Find the slope of the tangent line to the cardioid \( r = 1 + \sin \theta \) at the point where \( \theta = \pi/3 \).

Solution
Example 2.22

Find the points on the cardioid \( r = 1 - \cos \theta \) at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution

Theorem 2.2

If the polar curve \( r = f(\theta) \) passes through the origin at \( \theta = \theta_0 \), and if \( \frac{dr}{d\theta} \neq 0 \) at \( \theta = \theta_0 \), then the line \( \theta = \theta_0 \) is tangent to the curve at the origin.

Arc Length for Polar Curves

The formula for the length of a polar curve can be obtained from the arc length formula for a curve described by parametric equations.

Theorem 2.3 (Arc Length of a Polar Curve)

Let \( f \) be a function whose derivative is continuous on an interval \( \alpha \leq \theta \leq \beta \). The length of the graph of \( r = f(\theta) \) from \( \theta = \alpha \) to \( \theta = \beta \) is

\[
L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]

(2.6)

Note: When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration.
Example 2.23
Find the arc length of the spiral \( r = e^\theta \) between \( \theta = 0 \) and \( \theta = \pi \).

Solution

Example 2.24
Find the total arc length of the cardioid \( r = 1 + \cos \theta \).

Solution

Area in Polar Coordinates
In this section we develop the formula for the area of a region whose boundary is given by a polar equation.

Suppose that \( \alpha \) and \( \beta \) are angles that satisfy the condition
\[
\alpha < \beta \leq \alpha + 2\pi.
\]
If \( f(\theta) \) is continuous and either nonnegative or nonpositive for \( \alpha \leq \theta \leq \beta \), then the area \( A \) of the region \( R \) bounded by the polar curve \( r = f(\theta) \) and the lines \( \theta = \alpha \) and \( \theta = \beta \) is

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta
\]

(2.7)

The hardest part of applying (2.7) is determining the limits of integration. This can be done as follows:

**Step 1** Sketch the region \( R \) whose area is to be determined.

**Step 2** Draw an arbitrary “radial line” from the pole to the boundary curve \( r = f(\theta) \).

**Step 3** Ask, “Over what intervals must \( \theta \) vary in order for the radial line to sweep out the region \( R \)?”

**Step 4** The answer in Step 3 will determine the lower and upper limits of integration.

**Example 2.25**

Find the area of the region in the first quadrant that is within the cardioid \( r = 1 - \cos \theta \).

**Solution**

**Example 2.26**

Find the area enclosed by the rose curve \( r = \cos 2\theta \).

**Solution**
Let \( R \) be the region bounded by curves with polar equations \( r = f(\theta) \), \( r = g(\theta) \), \( \theta = \alpha \), and \( \theta = \beta \), where \( f(\theta) \geq g(\theta) \) and \( \alpha < \beta \leq \alpha + 2\pi \).

Then the area \( A \) of \( R \) is

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} \left( [f(\theta)]^2 - [g(\theta)]^2 \right) d\theta
\]

**Example 2.27**

Find the area of the region that is inside of the cardioid \( r = 4 + 4 \cos \theta \) and outside of the circle \( r = 6 \).

**Solution**

**Exercise 2.3**

1. \( r = 2 \sin \theta; \ \theta = \pi/6 \)
2. \( r = 1 + \cos \theta; \ \theta = \pi/2 \)
3. \( r = 1/\theta; \ \theta = 2 \)
4. \( r = a \sec 2\theta; \ \theta = \pi/6 \)
5. \( r = \sin 3\theta; \ \theta = \pi/4 \)
6. \( r = 4 - 3 \sin \theta; \ \theta = \pi \)
Find polar coordinates of all points at which the polar curve has a horizontal or a vertical tangent line.

6. \( r = a(1 + \cos \theta) \)  
7. \( r = a \sin \theta \)

Use Formula (2.6) to calculate the arc length of the polar curve.

8. The entire circle \( r = a \)
9. The entire circle \( r = 2a \cos \theta \)
10. The entire cardioid \( r = a(1 - \cos \theta) \)
11. \( r = \sin^2(\theta/2) \) from \( \theta = 0 \) to \( \theta = \pi \)
12. \( r = e^{3\theta} \) from \( \theta = 0 \) to \( \theta = 2 \)
13. \( r = \sin^3(\theta/3) \) from \( \theta = 0 \) to \( \theta = \pi/2 \)

In each part, find the area of the circle by integration.

(a) \( r = 2a \sin \theta \)  
(b) \( r = 2a \cos \theta \)

Find the area of the region described.

15. The region that is enclosed by the cardioid \( r = 2 + 2 \sin \theta \).
16. The region in the first quadrant within the cardioid \( r = 1 + \cos \theta \).
17. The region enclosed by the rose \( r = 4 \cos 3\theta \).
18. The region enclosed by the rose \( r = 2 \sin 2\theta \).
19. The region inside the circle \( r = 3 \sin \theta \) and outside the cardioid \( r = 1 + \sin \theta \).
20. The region outside the cardioid \( r = 2 - 2 \cos \theta \) and inside the circle \( r = 4 \).
21. The region inside the cardioid \( r = 2 + 2 \cos \theta \) and outside the circle \( r = 3 \).
22. The region inside the rose \( r = 2a \cos 2\theta \) and outside the circle \( r = a\sqrt{2} \).

Answers to Exercise 2.3

1. \( \sqrt{3} \)  
2. \( \frac{\tan 2 - 2}{2\tan 2 + 1} \)  
3. \( \frac{1}{2} \)  
4. \( 1/2 \)  
5. \( 5 \)
6. horizontal: \( (3a/2, \pi/3), (0, \pi), (3a/2, 5\pi/3) \)  
vertical: \( (2a, 0), (a/2, 2\pi/3), (a/2, 4\pi/3) \)
8. \( L = 2\pi a \)  
10. \( L = 8a \)  
14. (a) \( \pi a^2 \)  
(b) \( \pi a^2 \)  
15. \( 6\pi \)  
17. \( 4\pi \)  
18. \( \pi \)  
21. \( 9\sqrt{3}/2 - \pi \)