Chapter 6

Inner Product Spaces

6.1 Inner Products

**Definition 6.1.**

An *inner product* on a real vector space $V$ is an operator on $V$ that assigns to each pair of vector $u$ and $v$ in $V$ a real number $\langle u, v \rangle$ satisfying the following conditions:

(i) $\langle u, v \rangle = \langle v, u \rangle$ for all $u$ and $v$ in $V$.

(ii) $\langle u + v, z \rangle = \langle u, z \rangle + \langle v, z \rangle$ for all $u$, $v$, $z$ in $V$.

(iii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ for all $u$, $v$ in $V$ and all scalars $\alpha$.

(iv) $\langle u, u \rangle \geq 0$ with equality if and only if $u = 0$.

A real vector space $V$ with an inner product is called a *real inner product space*.

**Remark** In Section 6.4 we shall study inner products over complex vector spaces. However, until that time we shall use the term “inner product space” to mean real inner product space.
If \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) and \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) are vectors in \( \mathbb{R}^n \), then the standard inner product for \( \mathbb{R}^n \) is defined by

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n
\]

This inner product is also known as the \textit{Euclidean inner product on} \( \mathbb{R}^n \).

Given a vector \( \mathbf{w} \) with positive entries, we could also define an inner product on \( \mathbb{R}^n \) by

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} w_i u_i v_i
\]

the entries \( w_i \) are referred to as \textit{weights}. ♠

Example 6.1.

Let \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \) be vectors in \( \mathbb{R}^2 \). Verify that the weighted Euclidean inner product

\[
\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2
\]

satisfies the four inner product conditions.

Solution
Example 6.3.

Given $A$ and $B$ in $M_{m \times n}$, we can define an inner product on $M_{m \times n}$ by

$$
\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} \quad (6.1)
$$

Example 6.4.

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers. For each pair of polynomials in $P_{n-1}$, define

$$
\langle p, q \rangle = \sum_{i=1}^{n} p(x_i)q(x_i) \quad (6.2)
$$

It is easily seen that (6.2) satisfies conditions (i), (ii), and (iii) of the definition of an inner product. To show that (iv) holds, note that

$$
\langle p, p \rangle = \sum_{i=1}^{n} (p(x_i))^2 \geq 0
$$

If $\langle p, p \rangle = 0$, then $x_1, x_2, \ldots, x_n$ must be roots of $p(x) = 0$. Since $p(x)$ is of degree less than or equal $n - 1$, it must be the zero polynomial.

If $w(x)$ is a positive function, then

$$
\langle p, q \rangle = \sum_{i=1}^{n} w(x_i)p(x_i)q(x_i)
$$

also defines an inner product on $P_{n-1}$.  

Example 6.5.

Let $f = f(x)$ and $g = g(x)$ be two functions in $C[-1, 1]$, the vector space of all real-valued functions continuous on the interval $[-1, 1]$. Define

$$
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx
$$

Verify that $\langle f, g \rangle$ is an inner product on $C[-1, 1]$.

Solution
Length and Distance in Inner Product Spaces

**Definition 6.2.**

If $V$ is an inner product space, then the **norm** (or **length**) of a vector $v$ in $V$ is denoted by $\|v\|$ and is defined by

$$\|v\| = \langle v, v \rangle^{1/2} = \sqrt{\langle v, v \rangle}$$

The **distance** between two vectors $u$ and $v$ is denoted by $d(u, v)$ and is defined by

$$d(u, v) = \|u - v\|$$

If a vector has norm 1, then we say that it is a **unit vector**.

**Example 6.6.**

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|u\| = \langle u, u \rangle^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

and

$$d(u, v) = \|u - v\| = \langle u - v, u - v \rangle^{1/2} = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

For example, for the vectors $u = (1, 0)$ and $v = (0, 1)$ in $\mathbb{R}^2$ with the Euclidean inner product, we have

$$\|u\| = \sqrt{1^2 + 0^2} = 1$$

and

$$d(u, v) = \|u - v\| = \|(1, -1)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

However, if we change to the weighted Euclidean inner product in Example 6.2,

$$\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$$

then we obtain

$$\|u\| = \langle u, u \rangle^{1/2} = \sqrt{3(1)(1) + 2(0)(0)} = \sqrt{3}$$

and

$$d(u, v) = \|u - v\| = \langle u - v, u - v \rangle^{1/2} = \langle (1, -1), (1, -1) \rangle^{1/2} = \sqrt{3(1)(1) + 2(-1)(-1)} = \sqrt{5}$$
Inner Products Generated by Matrices

The Euclidean inner product and the weighted Euclidean inner product are special cases of a general class of inner products on \( \mathbb{R}^n \), which we shall now describe. Let

\[
    \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
\]

be vectors in \( \mathbb{R}^n \), and let \( A \) be an invertible \( n \times n \) matrix. It can be shown that if \( \mathbf{u} \cdot \mathbf{v} \) is the Euclidean inner product on \( \mathbb{R}^n \), then the formula

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} \quad (6.3)
\]

defines an inner product; it is called the **inner product on** \( \mathbb{R}^n \) **generated by** \( A \).

Moreover, (6.3) can be written in the alternative form

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u})^T A\mathbf{v}
\]

or, equivalently,

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A^T A\mathbf{v} \quad (6.4)
\]

**Example 6.7.**

The inner product on \( \mathbb{R}^n \) generated by the \( n \times n \) identity matrix is the Euclidean inner product, since substituting \( A = I \) in (6.3) yields

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}
\]

The weighted Euclidean inner product \( \langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 \) discussed in Example 6.2 is the inner product on \( \mathbb{R}^2 \) generated by

\[
    A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}
\]

because substituting this matrix in (6.4) yields

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 3u_1v_1 + 2u_2v_2
\]

In general, the weighted Euclidean inner product

\[
    \langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n
\]
is the inner product on $\mathbb{R}^n$ generated by

$$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{w_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix}$$

Example 6.8.

If

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

are any two $2 \times 2$ matrices, then the following formula defines an inner product on $M_{2 \times 2}$:

$$\langle A, B \rangle = \text{tr}(A^T B) = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle A, B \rangle = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

The norm of a matrix $A$ relative to this inner product is

$$\|A\| = \langle A, A \rangle^{1/2} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \quad \clubsuit$$

Theorem 6.1 (Properties of Inner Products).

If $u$, $v$, and $w$ are vectors in a inner product space, and $k$ is any scalar, then

(a) $\langle 0, v \rangle = \langle v, 0 \rangle = 0$

(b) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

(c) $\langle u, kv \rangle = k \langle u, v \rangle$

(d) $\langle u - v, w \rangle = \langle u, w \rangle - \langle v, w \rangle$

(e) $\langle u, v - w \rangle = \langle u, v \rangle - \langle u, w \rangle$
Example 6.9.

\[ \langle u - 2v, 3u + 4v \rangle = \langle u, 3u + 4v \rangle - \langle 2v, 3u + 4v \rangle \]
\[ = \langle u, 3u \rangle + \langle u, 4v \rangle - \langle 2v, 3u \rangle - \langle 2v, 4v \rangle \]
\[ = 3\langle u, u \rangle + 4\langle u, v \rangle - 6\langle v, u \rangle - 8\langle v, v \rangle \]
\[ = 3\|u\|^2 + 4\langle u, v \rangle - 6\langle v, u \rangle - 8\|v\|^2 \]
\[ = 3\|u\|^2 - 2\langle u, v \rangle - 8\|v\|^2 \]

Exercise 6.1

1. Let \( \langle u, v \rangle \) be the Euclidean inner product on \( \mathbb{R}^2 \), and let \( u = (3, -2), \ v = (4, 5), \ w = (-1, 6) \), and \( \alpha = -4 \). Verify that
   (a) \( \langle u, v \rangle = \langle v, u \rangle \)
   (b) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \)
   (c) \( \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \)
   (d) \( \langle \alpha u, v \rangle = \alpha \langle u, v \rangle = \langle u, \alpha v \rangle \)
   (e) \( \langle 0, v \rangle = \langle v, 0 \rangle = 0 \)

2. Repeat Exercise 1 for the weighted Euclidean inner product
   \[ \langle u, v \rangle = 4u_1v_1 + 5u_2v_2. \]

3. Compute \( \langle u, v \rangle \) using the inner product in Example 6.8.
   (a) \( u = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 8 \end{bmatrix}, \ v = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \)
   (b) \( u = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix}, \ v = \begin{bmatrix} 4 \\ 6 \\ 0 \\ 8 \end{bmatrix} \)

4. (a) Use Formula (6.3) to show that \( \langle u, v \rangle = 9u_1v_1 + 4u_2v_2 \) is the inner product on \( \mathbb{R}^2 \) generated by
   \[ A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \]
   (b) Use the inner product in part (a) to compute \( \langle u, v \rangle \) if \( u = (-3, 2) \) and \( v = (1, 7) \).

5. Let \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). In each part, the given expression is an inner product on \( \mathbb{R}^2 \). Find a matrix that generates it.
   (a) \( \langle u, v \rangle = 3u_1v_1 + 5u_2v_2 \)
   (b) \( \langle u, v \rangle = 4u_1v_1 + 6u_2v_2 \)

6. Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \). Determine which of the following are inner products on \( \mathbb{R}^3 \). For those that are not, list the conditions that do not hold.
   (a) \( \langle u, v \rangle = u_1v_1 + u_3v_3 \)
   (b) \( \langle u, v \rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 \)
   (c) \( \langle u, v \rangle = 2u_1v_1 + u_2v_2 + 4u_3v_3 \)
   (d) \( \langle u, v \rangle = u_1v_1 - u_2v_2 + u_3v_3 \)
7. In each part, use the given inner product on \( \mathbb{R}^2 \) to find \( d(u, v) \) for \( u = (-1, 2) \) and \( v = (2, 5) \).

(a) the Euclidean inner product
(b) the weighted Euclidean inner product \( \langle u, v \rangle = 3u_1v_1 + 2u_2v_2 \), where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \)
(c) the inner product generated by the matrix

\[
A = \begin{bmatrix}
1 & 2 \\
-1 & 3
\end{bmatrix}
\]

8. Let \( M_{2 \times 2} \) have the inner product in Example 6.8. In each part, find \( \|A\| \).

(a) \( A = \begin{bmatrix}
-2 & 5 \\
3 & 6
\end{bmatrix} \)
(b) \( A = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \)

9. Let \( M_{2 \times 2} \) have the inner product in Example 6.8. Find \( d(A, B) \).

(a) \( A = \begin{bmatrix}
2 & 6 \\
9 & 4
\end{bmatrix} \), \( B = \begin{bmatrix}
-4 & 7 \\
1 & 6
\end{bmatrix} \)
(b) \( A = \begin{bmatrix}
-2 & 4 \\
1 & 0
\end{bmatrix} \), \( B = \begin{bmatrix}
-5 & 1 \\
6 & 2
\end{bmatrix} \)

10. Let the vector space \( P_2 \) have the inner product

\[
\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx
\]

(a) Find \( \|p\| \) for \( p = 1, p = x, \) and \( p = x^2 \).
(b) Find \( d(p, q) \) if \( p = 1 \) and \( q = x \).

11. Let \( A = \begin{bmatrix}
a_1 & a_2 \\
a_3 & a_4
\end{bmatrix} \) and \( B = \begin{bmatrix}
b_1 & b_2 \\
b_3 & b_4
\end{bmatrix} \).

Show that \( \langle A, B \rangle = a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4 \) is not an inner product on \( M_{2 \times 2} \).

12. Use the inner product

\[
\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx
\]

to compute \( \langle p, q \rangle \) for the vectors \( p = p(x) \) and \( q = q(x) \) in \( P_3 \).

(a) \( p = 1 - x + x^2 + 5x^3, \quad q = x - 3x^2 \)
(b) \( p = x - 5x^3, \quad q = 2 + 8x^2 \)

**Answer to Selected Exercise 6.1**

1. (a) 2 (b) 11 (c) -13 (d) -8 (e) 0 3. (a) 3 (b) 56 4. (b) 29
5. (a) \( \begin{bmatrix}
\sqrt{3} & 0 \\
0 & \sqrt{5}
\end{bmatrix} \) (b) \( \begin{bmatrix}
2 & 0 \\
0 & \sqrt{6}
\end{bmatrix} \)
6. (a) No. Condition 4 fails.  (b) No. Conditions 2 and 3 fail.  (c) Yes.
   (d) No. Condition 4 fails.

7. (a) $3\sqrt{2}$  (b) $3\sqrt{5}$  (c) $3\sqrt{13}$  
8. (a) $\sqrt{74}$  (b) 0  
9. (a) $\sqrt{105}$  (b) $\sqrt{47}$

10. (a) $\sqrt{2}, \frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{10}$  (b) $\frac{2}{3}\sqrt{6}$  
12. (a) $-\frac{28}{15}$  (b) 0

### 6.2 Angle and Orthogonality in Inner Product Spaces

Recall that if $u$ and $v$ are nonzero vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and $\theta$ is the angle between them, then

$$ u \cdot v = \|u\| \|v\| \cos \theta $$

or, alternatively,

$$ \cos \theta = \frac{u \cdot v}{\|u\| \|v\|} \quad (6.5) $$

Our first goal in this section is to define the concept of an angle between two vectors in a general inner product space. For such a definition to be reasonable, we would want it to be consistent with Formula (6.5) when it applied to the special case of $\mathbb{R}^2$ and $\mathbb{R}^3$ with the Euclidean inner product.

Since $|\cos \theta \leq 1|$, every pair of nonzero vectors in an inner product satisfies the inequality

$$ \left| \frac{u \cdot v}{\|u\| \|v\|} \right| \leq 1 $$

**Theorem 6.2 (Cauchy-Schwarz Inequality).**

If $u$ and $v$ are vectors in a real inner product space, then

$$ |\langle u, v \rangle| \leq \|u\| \|v\| $$

For reference, we note that the Cauchy-Schwarz Inequality can be written in the following two alternative forms:

$$ \langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle \quad (6.6) $$

$$ \langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2 \quad (6.7) $$


**Theorem 6.3 (Properties of Length).**

If \( u \) and \( v \) are vectors in an inner product space \( V \), and if \( k \) is any scalar, then

(a) \( \| u \| \geq 0 \)

(b) \( \| u \| = 0 \) if and only if \( u = 0 \)

(c) \( \| ku \| = |k| \| u \| \)

(d) \( \| u + v \| \leq \| u \| + \| v \| \) \quad \text{(Triangle inequality)}

**Theorem 6.4 (Properties of Distance).**

If \( u, v, \) and \( w \) are vectors in an inner product space \( V \), and if \( k \) is any scalar, then

(a) \( d(u, v) \geq 0 \)

(b) \( d(u, v) = 0 \) if and only if \( u = v \)

(c) \( d(u, v) = d(v, u) \)

(d) \( d(u, v) \leq d(u, w) + d(w, v) \) \quad \text{(Triangle inequality)}

**Angle Between Vectors**

We shall now show how the Cauchy-Schwarz inequality can be used to define angles in general inner product spaces. Suppose that \( u \) and \( v \) are nonzero vectors in an inner product space \( V \). If we divide both sides of Formula (6.7) by \( \| u \|^2 \| v \|^2 \), we obtain

\[
\left( \frac{\langle u, v \rangle}{\| u \| \| v \|} \right)^2 \leq 1
\]

or, equivalently,

\[
-1 \leq \frac{\langle u, v \rangle}{\| u \| \| v \|} \leq 1 \quad (6.8)
\]

Thus, from (6.8), there is a unique angle \( \theta \) such that

\[
\cos \theta = \frac{\langle u, v \rangle}{\| u \| \| v \|} \quad \text{and} \quad 0 \leq \theta \leq \pi \quad (6.9)
\]

We define \( \theta \) to be the **angle between \( u \) and \( v \)**.

**Example 6.10.**

Let \( \mathbb{R}^4 \) have the Euclidean inner product. Find the cosine of the angle \( \theta \) between the vectors \( u = (4, 3, 1, -2) \) and \( v = (-2, 1, 2, 3) \).

**Solution**
Orthogonality

**Definition 6.3.**
Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in an inner product are called **orthogonal** if \( \langle \mathbf{u}, \mathbf{v} \rangle = 0 \).

**Example 6.11.**
If \( M_{2 \times 2} \) has the inner product of Example 6.8 in the preceding section, then the matrices
\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}
\]
are orthogonal, since
\[
\langle A, B \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0
\]

**Example 6.12.**
Let \( P_2 \) have the inner product
\[
\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x) \, dx
\]
and let \( \mathbf{p} = x \) and \( \mathbf{q} = x^2 \). Then
\[
\| \mathbf{p} \| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left( \int_{-1}^{1} xx \, dx \right)^{1/2} = \left( \int_{-1}^{1} x^2 \, dx \right)^{1/2} = \sqrt{\frac{2}{3}}
\]
\[
\| \mathbf{q} \| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left( \int_{-1}^{1} x^2 x^2 \, dx \right)^{1/2} = \left( \int_{-1}^{1} x^4 \, dx \right)^{1/2} = \sqrt{\frac{2}{5}}
\]
\[
\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2 \, dx = \int_{-1}^{1} x^3 \, dx = 0
\]
Because \( \langle \mathbf{p}, \mathbf{q} \rangle = 0 \), the vectors \( \mathbf{p} = x \) and \( \mathbf{q} = x^2 \) are orthogonal relative to the given inner product.

**Theorem 6.5 (Generalized Theorem of Pythagoras).**
If \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal vectors in an inner product space, then
\[
\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2
\]
Example 6.13.

In Example 6.12 we showed that \( p = x \) and \( q = x^2 \) are orthogonal relative to the inner product

\[
\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx
\]

on \( P_2 \). It follows from the Theorem of Pythagoras that

\[
\|p + q\|^2 = \|p\|^2 + \|q\|^2
\]

Thus, from the computations in Example 6.12, we have

\[
\|p + q\|^2 = \left( \sqrt{\frac{2}{3}} \right)^2 + \left( \sqrt{\frac{2}{5}} \right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}
\]

We can check this result by direct integration:

\[
\|p + q\|^2 = \langle p + q, p + q \rangle = \int_{-1}^{1} (x + x^2)(x + x^2) \, dx = \int_{-1}^{1} x^2 \, dx + 2 \int_{-1}^{1} x^3 \, dx + \int_{-1}^{1} x^4 \, dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}
\]

Orthogonal Complements

If \( V \) is a plane though the origin of \( \mathbb{R}^3 \) with the Euclidean inner product, then the set of all vectors that are orthogonal to every vector in \( V \) forms the line \( L \) through the origin that is perpendicular to \( V \). In the language of linear algebra we say that the line and the plane are orthogonal complements of one another. The following definition extends this concept to general inner product spaces.

![Diagram showing orthogonal complements](image-url)
Definition 6.4.

Let $W$ be a subspace of an inner product space $V$. A vector $u$ in $V$ is said to be **orthogonal** to $W$ if it is orthogonal to every vector in $W$, and the set of all vectors in $V$ that are orthogonal to $W$ is called the **orthogonal complement of $W$** and is denoted by $W^\perp$.

Theorem 6.6 (Properties of Orthogonal Complements).

If $W$ is a subspace of a finite-dimensional inner product space $V$, then

(a) $W^\perp$ is a subspace of $V$.

(b) The only vector common to $W$ and $W^\perp$ is 0.

(c) The orthogonal complement of $W^\perp$ is $W$; that is, $(W^\perp)^\perp = W$.

Remark

Because $W$ and $W^\perp$ are orthogonal complements of one another by part (c) of the Theorem 6.6, we shall say that $W$ and $W^\perp$ are **orthogonal complements**.

Theorem 6.7.

If $A$ is an $m \times n$ matrix, then

(a) The nullspace of $A$ and the row space of $A$ are orthogonal complements in $\mathbb{R}^n$ with respect to the Euclidean inner product.

(b) The nullspace of $A^T$ and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$ with respect to the Euclidean inner product.


Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

\[ w_1 = (1, -1, 2, 1) \quad w_2 = (0, 1, 1, -2) \quad w_3 = (1, -3, 0, 5) \]

Find a basis for the orthogonal complement of $W$.

**Solution**
Orthonormal

**Definition 6.5.**

A set of vectors in an inner product space is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is called **orthonormal**.

**Example 6.15.**

Let

\[ u_1 = (0, 1, 0), \quad u_2 = (1, 0, 1), \quad u_3 = (1, 0, -1) \]

and assume that \( \mathbb{R}^3 \) has the Euclidean inner product. It follows that the set of vectors \( S = \{u_1, u_2, u_3\} \) is orthogonal since \( \langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0 \).

If \( v \) is a nonzero vector in an inner product space, then by part (c) of Theorem 6.3, the vector

\[ \frac{1}{||v||} v \]

has norm 1, since

\[ \left\| \frac{1}{||v||} v \right\| = \frac{1}{||v||} \left\| v \right\| = \frac{1}{||v||} ||v|| = 1 \]

The process of multiplying a nonzero vector \( v \) by the reciprocal of its length to obtain a unit vector is called **normalizing** \( v \). An orthogonal set of nonzero vectors can always be converted to an orthonormal set by normalizing each of its vectors.

**Example 6.16.**

The Euclidean norm of the vectors in Example 6.15 are

\[ ||u_1|| = 1, \quad ||u_2|| = \sqrt{2}, \quad ||u_3|| = \sqrt{2} \]

Consequently, normalizing \( u_1, u_2, \) and \( u_3 \) yields

\[ v_1 = \frac{u_1}{||u_1||} = (0, 1, 0), \quad v_2 = \frac{u_2}{||u_2||} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \]

\[ v_3 = \frac{u_3}{||u_3||} = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \]

We can verify that the set \( S = \{v_1, v_2, v_3\} \) are orthonormal by showing that

\[ \langle v_1, v_2 \rangle = \langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0 \quad \text{and} \quad ||v_1|| = ||v_2|| = ||v_3|| = 1 \]
In an inner product space, a basis consisting of orthonormal vectors is called an \textbf{orthonormal basis}, and a basis consisting of orthogonal vectors is called an \textbf{orthogonal basis}. A familiar example of an orthonormal basis is the standard basis for $\mathbb{R}^3$ with the Euclidean inner product:

$$i = (1, 0, 0), \quad j = (0, 1, 0), \quad k = (0, 0, 1)$$

More generally, in $\mathbb{R}^n$ with the Euclidean inner product, the standard basis

$$e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_n = (0, 0, 0, \ldots, 1)$$

is orthonormal.

\textbf{Coordinates Relative to Orthonormal Bases}

**Theorem 6.8.**

If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space $V$ and $u$ is any vector in $V$, then

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots + \langle u, v_n \rangle v_n$$

**Example 6.17.**

Let

$$v_1 = (0, 1, 0), \quad v_2 = \left(\frac{4}{5}, 0, \frac{3}{5}\right), \quad v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

It is easy to check that $S = \{v_1, v_2, v_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product. Express the vector $u = (1, 1, 1)$ as a linear combination of the vectors in $S$, and find the coordinate vector $(u)_S$.

\textbf{Solution}
If $S$ is an orthonormal basis for an $n$-dimensional inner product space, and if 

$$(u)_S = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad (v)_S = (v_1, v_2, \ldots, v_n)$$

then

(a) $\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

(b) $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$

(c) $\langle u, v \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

**Theorem 6.9.**

If $\mathbb{R}^3$ has the Euclidean inner product, then the norm of the vector $u = (1, 1, 1)$ is 

$$\|u\| = (u \cdot u)^{1/2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

However, if we let $\mathbb{R}^3$ have the orthonormal basis $S$ in Example 6.17, then we know from that example that the coordinate vector of $u$ relative to $S$ is 

$$(u)_S = (1, -\frac{1}{5}, \frac{7}{5})$$

The norm of $u$ can also be calculated from this vector using part (a) of Theorem 6.9. This yields 

$$\|u\| = \sqrt{1^2 + \left(-\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{75}{25}} = \sqrt{3} \quad \blacklozenge$$

**Example 6.18.**

If $\mathbb{R}^3$ has the Euclidean inner product, then the norm of the vector $u = (1, 1, 1)$ is 

$$\|u\| = (u \cdot u)^{1/2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

However, if we let $\mathbb{R}^3$ have the orthonormal basis $S$ in Example 6.17, then we know from that example that the coordinate vector of $u$ relative to $S$ is 

$$(u)_S = (1, -\frac{1}{5}, \frac{7}{5})$$

The norm of $u$ can also be calculated from this vector using part (a) of Theorem 6.9. This yields 

$$\|u\| = \sqrt{1^2 + \left(-\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{75}{25}} = \sqrt{3} \quad \blacklozenge$$

**Coordinates Relative to Orthogonal Bases**

If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis for a vector space $V$, then normalizing each of these vectors yields the orthonormal basis 

$$S' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}$$

Thus, if $u$ is any vector in $V$, it follow from Theorem 6.8 that 

$$u = \left\langle u, \frac{v_1}{\|v_1\|} \right\rangle \frac{v_1}{\|v_1\|} + \left\langle u, \frac{v_2}{\|v_2\|} \right\rangle \frac{v_2}{\|v_2\|} + \cdots + \left\langle u, \frac{v_n}{\|v_n\|} \right\rangle \frac{v_n}{\|v_n\|}$$

which, by part (c) of Theorem 6.3, can be written as 

$$u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_n \rangle}{\|v_n\|^2} v_n \quad (6.10)$$

This formula expresses $u$ as a linear combination of the vectors in the orthogonal basis $S$. 
Theorem 6.10.

If \( S = \{v_1, v_2, \ldots, v_n\} \) is an orthogonal set of nonzero vectors in an inner product space, then \( S \) is linearly independent.

Orthogonal Projections

In \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) with the Euclidean inner product, it is evident geometrically that if \( W \) is a line or a plane through the origin, then each vector \( u \) in the space can be expressed as a sum

\[
u = w_1 + w_2
\]

where \( w_1 \) is in \( W \) and \( w_2 \) is perpendicular to \( W \).

Theorem 6.11 (Projection Theorem).

If \( W \) is a finite-dimensional subspace of an inner product space \( V \), then every vector \( u \) in \( V \) can be expressed in exactly one way as

\[
u = w_1 + w_2
\]

where \( w_1 \) is in \( W \) and \( w_2 \) is in \( W^\perp \).

The vector \( w_1 \) in Theorem 6.11 is called the **orthogonal projection of \( u \) on \( W \)** and is denoted by \( \text{proj}_W u \). The vector \( w_2 \) is called the **component of \( u \) orthogonal to \( W \)** and is denoted by \( \text{proj}_{W^\perp} u \). Thus Formula in the Projection Theorem can be expressed as

\[
u = \text{proj}_W u + \text{proj}_{W^\perp} u
\]  

(6.11)

Since \( w_2 = u - w_1 \), it follows that

\[
\text{proj}_{W^\perp} u = u - \text{proj}_W u
\]

so Formula (6.11) can also be written as

\[
u = \text{proj}_W u + (u - \text{proj}_W u)
\]  

(6.12)
Let $W$ be a finite-dimensional subspace of an inner product space $V$.

(a) If $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis for $W$, and $u$ is any vector in $V$, then

$$\text{proj}_W u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots + \langle u, v_r \rangle v_r$$

(6.13)

(b) If $\{v_1, v_2, \ldots, v_r\}$ is an orthogonal basis for $W$, and $u$ is any vector in $V$, then

$$\text{proj}_W u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_r \rangle}{\|v_r\|^2} v_r$$

(6.14)

Example 6.19.

Let $\mathbb{R}^3$ have the Euclidean inner product, and let $W$ be the subspace spanned by the orthonormal vectors $v_1 = (0, 1, 0)$ and $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$. From (6.13) the orthogonal projection of $u = (1, 1, 1)$ on $W$ is

$$\text{proj}_W u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 = (1)(0, 1, 0) + \left(-\frac{1}{5}\right) \left(-\frac{4}{5}, 0, \frac{3}{5}\right) = \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

The component of $u$ orthogonal to $W$ is

$$\text{proj}_{W^\perp} u = u - \text{proj}_W u = (1, 1, 1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right) = \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

Observe that $\text{proj}_{W^\perp} u$ is orthogonal to both $v_1$ and $v_2$, so this vector is orthogonal to each vector in the space $W$ spanned by $v_1$ and $v_2$, as it should be.

Exercise 6.2

1. In each part, determine whether the given vectors are orthogonal with respect to the Euclidean inner product.
   (a) $u = (-1, 3, 2), \ v = (4, 2, -1)$
   (b) $u = (-2, -2, -2), \ v = (1, 1, 1)$
   (c) $u = (u_1, u_2, u_3), \ v = (0, 0, 0)$
   (d) $u = (a, b), \ v = (-b, a)$
   (e) $u = (0, 3, -2, 1), \ v = (5, 2, -1, 0)$
   (f) $u = (-4, 6, -10, 1), \ v = (2, 1, -2, 9)$

2. Let $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ have the Euclidean inner product. In each part, find the cosine of the angle between $u$ and $v$. 
(a) \( \mathbf{u} = (1, -3), \; \mathbf{v} = (2, 4) \)  
(b) \( \mathbf{u} = (-1, 0), \; \mathbf{v} = (3, 8) \)  
(c) \( \mathbf{u} = (-1, 5, 2), \; \mathbf{v} = (2, 4, -9) \)  
(d) \( \mathbf{u} = (4, 1, 8), \; \mathbf{v} = (1, 0, -3) \)  
(e) \( \mathbf{u} = (1, 0, 1, 0), \; \mathbf{v} = (-3, -3, -3, -3) \)  
(f) \( \mathbf{u} = (2, 1, 7, -1), \; \mathbf{v} = (4, 0, 0, 0) \)

3. Let 
\[
A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}
\]

Which of the following matrices are orthogonal to \( A \) with respect to the inner product in Example 6.8 of Section 6.1?

(a) \[
\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}
\]
(b) \[
\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}
\]
(c) \[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
(d) \[
\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}
\]

4. Let \( \mathbb{R}^4 \) have the Euclidean inner product. Find two unit vectors that are orthogonal to three vectors \( \mathbf{u} = (2, 1, -4, 0), (-1, -1, 2, 2), \) and \( \mathbf{w} = (3, 2, 5, 4) \).

5. Let 
\[
A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}
\]

(a) Find bases for the column space of \( A \) and nullspace of \( A^T \).
(b) Verify that every vector in the column space of \( A \) is orthogonal to every vector in the nullspace of \( A^T \).

6. Find a basis for the orthogonal complement of the subspace of \( \mathbb{R}^n \) spanned by the vectors.

(a) \( \mathbf{v}_1 = (1, -1, 3), \; \mathbf{v}_2 = (5, -4, -4), \; \mathbf{v}_3 = (7, -6, 2) \)  
(b) \( \mathbf{v}_1 = (2, 0, -1), \; \mathbf{v}_2 = (4, 0, -2) \)  
(c) \( \mathbf{v}_1 = (1, 4, 5, 2), \; \mathbf{v}_2 = (2, 1, 3, 0), \; \mathbf{v}_3 = (-1, 3, 2, 2) \)  
(d) \( \mathbf{v}_1 = (1, 4, 5, 6, 9), \; \mathbf{v}_2 = (3, -2, 1, 4, -1), \; \mathbf{v}_3 = (-1, 0, -1, -2, -1), \; \mathbf{v}_4 = (2, 3, 5, 7, 8) \)

7. Indicate whether each statement is always true or sometimes false.

(a) If \( V \) is a subspace of \( \mathbb{R}^n \) and \( W \) is a subspace of \( V \), then \( W^\perp \) is a subspace of \( V^\perp \).
(b) \( \| \mathbf{u} + \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \| + \| \mathbf{w} \| \) for all vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in an inner product space.
(c) If \( \mathbf{u} \) is in the row space and the nullspace of a square matrix \( A \), then \( \mathbf{u} = \mathbf{0} \).
(d) If \( \mathbf{u} \) is in the row space and the column space of an \( n \times n \) matrix \( A \), then \( \mathbf{u} = \mathbf{0} \).
8. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on \( \mathbb{R}^2 \)?
   (a) \((0, 1), (2, 0)\) 
   (b) \((-1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, 1/\sqrt{2})\) 
   (c) \((-1/\sqrt{2}, -1/\sqrt{2}), (1/\sqrt{2}, 1/\sqrt{2})\) 
   (d) \((0, 0), (0, 1)\)

9. Which of the following sets of vectors are orthonormal with respect to the Euclidean inner product on \( \mathbb{R}^3 \)?
   (a) \(\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\)
   (b) \(\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\)
   (c) \((1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (0, 0, 1)\)
   (d) \(\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\)

10. Which of the following sets of polynomials are orthonormal with respect to the inner product on \( P_3 \) defined by
    \[
    \langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2
    \]
    where \( p = a_0 + a_1x + a_2x^2 \) and \( q = b_0 + b_1x + b_2x^2 \).
    (a) \(\frac{2}{3} - \frac{2}{3}x + \frac{2}{3}x^2, \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2, \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2\)
    (b) \(1, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2, x^2\)

11. Verify that the given set of vectors is orthogonal with respect to the Euclidean inner product; then convert it to an orthonormal set by normalizing the vectors.
    (a) \((-1, 2), (6, 3)\)
    (b) \((1, 0, -1), (2, 0, 2), (0, 5, 0)\)
    (c) \(\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{5}\right), (-\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})\)

12. Verify that the vectors \( \mathbf{v}_1 = (-\frac{3}{7}, \frac{4}{7}, 0), \mathbf{v}_2 = (\frac{1}{5}, \frac{3}{5}, 0), \mathbf{v}_3 = (0, 0, 1) \) form an orthonormal basis for \( \mathbb{R}^3 \) with the Euclidean inner product; then use Theorem 6.8 to express each of the following as linear combinations of \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \)?
    (a) \((1, -1, 2)\) 
    (b) \((3, -7, 4)\) 
    (c) \(\left(\frac{1}{7}, -\frac{3}{7}, \frac{5}{7}\right)\)

13. In each part, an orthonormal basis relative to the Euclidean inner product is given. Use Theorem 6.8 to find the coordinate vector of \( \mathbf{w} \) with respect to that basis.
    (a) \( \mathbf{w} = (3, 7); \mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \)
    (b) \( \mathbf{w} = (-1, 0, 2); \mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \mathbf{u}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \)
14. Let $\mathbb{R}^3$ have the Euclidean inner product, and let $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the orthonormal basis with $\mathbf{w}_1 = (0, -\frac{3}{5}, \frac{4}{5})$, $\mathbf{w}_2 = (1, 0, 0)$, and $\mathbf{w}_3 = (0, \frac{4}{5}, \frac{3}{5})$.

(a) Find the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ that have the coordinate vectors $(\mathbf{u})_S = (-2, 1, 2)$, $(\mathbf{v})_S = (3, 0, -2)$, and $(\mathbf{w})_S = (5, -4, 1)$.

(b) Compute $||\mathbf{v}||$, $d(\mathbf{u}, \mathbf{w})$, and $(\mathbf{w}, \mathbf{v})$ by applying Theorem 6.9 to the coordinate vectors $(\mathbf{u})_S$, $(\mathbf{v})_S$, and $(\mathbf{w})_S$; then check the results by performing the computation directly on $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

15. (a) Show that the vectors $\mathbf{v}_1 = (1, -2, 3, -4)$, $\mathbf{v}_2 = (2, 1, -4, -3)$, $\mathbf{v}_3 = (-3, 4, 1, -2)$, and $\mathbf{v}_4 = (4, 3, 2, 1)$ form an orthogonal basis for $\mathbb{R}^4$ with the Euclidean inner product.

(b) Use (6.10) to express $\mathbf{u} = (-1, 2, 3, 7)$ as a linear combination of the vectors in part (a).

16. Find vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ that are orthonormal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ but are not orthonormal with respect to the Euclidean inner product.

**Answer to Exercise 6.2**

1. (a) Yes (b) No (c) Yes (d) Yes (e) No (f) No

2. (a) $-\frac{1}{\sqrt{2}}$ (b) $-\frac{3}{\sqrt{3}}$ (c) 0 (d) $-\frac{20}{9\sqrt{10}}$ (e) $-\frac{1}{\sqrt{2}}$ (f) $\frac{2}{\sqrt{55}}$

3. (a) Orthogonal (b) Orthogonal (c) Orthogonal (d) Not orthogonal

4. $\pm \frac{1}{\sqrt{5}} (-34, 44, -6, 11)$

5. (a) $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

6. (a) $(16, 19, 1)$ (b) $(0, 1, 0)$, $(\frac{1}{2}, 0, 1)$ (c) $(-1, -1, 1, 0)$, $(\frac{3}{7}, -\frac{4}{7}, 0, 1)$

(d) $(-1, -1, 1, 0)$, $(-1, -1, 0, 1)$, $(-1, -2, 0, 1)$

7. (a) False (b) True (c) True (d) False

8. (a), (b), (d) 9. (b), (d) 10. (a)

11. (a) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $(0, 1, 0)$

(c) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$

12. (a) $-\frac{7}{5} \mathbf{v}_1 + \frac{3}{5} \mathbf{v}_2 + 2 \mathbf{v}_3$ (b) $-\frac{37}{5} \mathbf{v}_1 - \frac{3}{5} \mathbf{v}_2 + 4 \mathbf{v}_3$ (c) $-\frac{3}{7} \mathbf{v}_1 - \frac{1}{7} \mathbf{v}_2 + \frac{5}{7} \mathbf{v}_3$

13. (a) $(\mathbf{w})_S = (-2\sqrt{2}, 5\sqrt{2})$ (b) $(\mathbf{w})_S = (0, -2, 1)$

14. (a) $\mathbf{u} = (1, \frac{14}{5}, -\frac{2}{5})$, $\mathbf{v} = (0, -\frac{17}{5}, \frac{6}{5})$, $\mathbf{w} = (-4, -\frac{11}{5}, \frac{23}{5})$

(b) $||\mathbf{v}|| = \sqrt{13}$, $d(\mathbf{u}, \mathbf{v}) = 5\sqrt{3}$, $\langle \mathbf{w}, \mathbf{v} \rangle = 13$

15. (b) $\mathbf{u} = -\frac{4}{5} \mathbf{v}_1 - \frac{11}{10} \mathbf{v}_2 + 0 \mathbf{v}_3 + \frac{1}{2} \mathbf{v}_4$

16. $(1/\sqrt{5}, 1/\sqrt{5})$, $(2/\sqrt{30}, -3/\sqrt{30})$
6.3 Least Squares Problems

In this section we shall show how orthogonal projections can be used to solve certain approximation problems. The results obtained in this section have a wide variety of applications in both mathematics and science.

Least Squares Solutions of Linear Systems

Up to now we have been concerned primarily with consistent systems of linear equations. In Chapter 1, we showed how a consistent linear system can be solved by the Gaussian elimination or Gauss-Jordan elimination. However, when a linear system $A\mathbf{x} = \mathbf{b}$ does not have a solution, one is interested in finding a best approximate solution. By an “approximate solution,” we mean a vector $\mathbf{x}$ such that it minimizes the value of $\|A\mathbf{x} - \mathbf{b}\|$ with respect to Euclidean inner product. The quantity $\|A\mathbf{x} - \mathbf{b}\|$ can be viewed as a measure of the “error”. If the system is consistent and $\mathbf{x}$ is an exact solution, then the error is zero, since $\|A\mathbf{x} - \mathbf{b}\| = \|\mathbf{0}\| = 0$. In general, the larger the value of $\|A\mathbf{x} - \mathbf{b}\|$, the more poorly $\mathbf{x}$ serves as an approximate solution of the system.

**Theorem 6.13 (Best Approximation Theorem).**

If $W$ is a finite-dimensional subspace of an inner product space $V$, and if $\mathbf{u}$ is a vector in $V$, then $\text{proj}_W \mathbf{u}$ is the **best approximation** to $\mathbf{u}$ from $W$ in the sense that

$$\|\mathbf{u} - \text{proj}_W \mathbf{u}\| < \|\mathbf{u} - \mathbf{w}\|$$

for every vector $\mathbf{w}$ in $W$ that is different from $\text{proj}_W \mathbf{u}$.

**Least Squares Problem:** Given a linear system $A\mathbf{x} = \mathbf{b}$ of $m$ equations in $n$ unknowns, find a vector $\mathbf{x}$, if possible, that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to Euclidean inner product on $\mathbb{R}^m$. Such a vector is called a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

**Remark** To understand the origin of the term least squares, let $\mathbf{e} = A\mathbf{x} - \mathbf{b}$, which we can view as the error vector the results from the approximation $\mathbf{x}$. If $\mathbf{e} = (e_1, e_2, \ldots, e_m)$, then a least squares solution minimizes $\|\mathbf{e}\| = (e_1^2 + e_2^2 + \cdots + e_m^2)^{1/2}$; hence it also minimizes $\|\mathbf{e}\|^2 = e_1^2 + e_2^2 + \cdots + e_m^2$. Hence the term least squares.

To solve the least squares problem, let $W$ be the column space of $A$. For each $n \times 1$ matrix $\mathbf{x}$, the product $A\mathbf{x}$ is a linear combination of the column vectors of $A$. Thus, as $\mathbf{x}$ varies over $\mathbb{R}^n$, the vector $A\mathbf{x}$ varies over all possible linear combinations of the column vectors of $A$; that is, $A\mathbf{x}$ varies over the entire column space $W$. Geometrically, solving the least squares problem amounts to finding a vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x}$ is the closest vector in $W$ to $\mathbf{b}$ (Figure 6.1).
Figure 6.1: A least squares solution $\mathbf{x}$ produces the vector $A\mathbf{x}$ in $W$ closest to $\mathbf{b}$.

It follows from the Best Approximation Theorem (6.13) that the closest vector in $W$ to $\mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ on $W$. Thus, for a vector $\mathbf{x}$ to be a least squares solution of $A\mathbf{x} = \mathbf{b}$, this vector must satisfy

$$A\mathbf{x} = \text{proj}_W \mathbf{b} \quad (6.15)$$

One could attempt to find least squares solutions of $A\mathbf{x} = \mathbf{b}$ by first calculating the vector $\text{proj}_W \mathbf{b}$ and then solving (6.15); however, there is a better approach. It follows from the Projection Theorem (6.11) and Formula (6.12) of Section 6.1 that

$$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \text{proj}_W \mathbf{b}$$

is orthogonal to $W$. But $W$ is the column space of $A$, so it follows from Theorem 6.7 that $\mathbf{b} - A\mathbf{x}$ lies in the nullspace of $A^T$. Therefore, a least squares solution of $A\mathbf{x} = \mathbf{b}$ must satisfy

$$A^T(\mathbf{b} - A\mathbf{x}) = 0$$

or, equivalently,

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

This is called the normal system associated with $A\mathbf{x} = \mathbf{b}$. Thus the problem of finding a least squares solution of $A\mathbf{x} = \mathbf{b}$ has been reduced to the problem of finding an exact solution of the associated normal system.

**Theorem 6.14.**

For any linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

is consistent, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$. Moreover, if $W$ is the column space of $A$, and $\mathbf{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of $\mathbf{b}$ on $W$ is

$$\text{proj}_W \mathbf{b} = A\mathbf{x}$$

**Uniqueness of Least Squares Solutions**

Before we examine some numerical example, we shall establish conditions under which a linear system is guaranteed to have a unique least squares solution. We shall need the following theorem.
Theorem 6.15.

If \( A \) is an \( m \times n \) matrix, then the following are equivalent.

(a) \( A \) has linearly independent column vectors.

(b) \( A^T A \) is invertible.

Theorem 6.16.

If \( A \) is an \( m \times n \) matrix with linearly independent column vectors, then for every \( m \times 1 \) matrix \( b \), the linear system \( A x = b \) has a unique least squares solution. This solution is given by

\[
x = (A^T A)^{-1} A^T b
\] (6.16)

Moreover, if \( W \) is the column space of \( A \), then the orthogonal projection of \( b \) on \( W \) is

\[
\text{proj}_W b = A x = A(A^T A)^{-1} A^T b
\] (6.17)

Example 6.20.

Find the least squares solution of the linear system \( A x = b \) given by

\[
\begin{align*}
x_1 - x_2 &= 4 \\
3x_1 + 2x_2 &= 1 \\
2x_1 + 4x_2 &= 3
\end{align*}
\]

and find the orthogonal projection of \( b \) on the column space of \( A \).

Solution
Example 6.21.

Find the least squares solution of $Ax = b$, where

$$A = \begin{bmatrix}
1 & 2 & -1 \\
2 & 3 & 1 \\
-1 & -1 & -2 \\
3 & 5 & 0
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}$$

Solution
Example 6.22.

Find the orthogonal projection of the vector \( \mathbf{u} = (-3, -3, 8, 9) \) on the subspace of \( \mathbb{R}^4 \) spanned by the vectors

\[
\mathbf{u}_1 = (3, 1, 0, 1), \quad \mathbf{u}_2 = (1, 2, 1, 1), \quad \mathbf{u}_3 = (-1, 0, 2, -1)
\]

Solution
1. Find the normal system associated with the given linear system.

(a) \[
\begin{bmatrix}
1 & -1 \\
2 & 3 \\
4 & 5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 & -1 & 0 \\
3 & 1 & 2 \\
-1 & 4 & 5 \\
1 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
0 \\
1 \\
2
\end{bmatrix}
\]

2. Find the least squares solution of the linear system \( Ax = b \), and find the orthogonal projection of \( b \) onto the column space of \( A \).

(a) \( A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \), \( b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \), \( b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \), \( b = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} \)

(d) \( A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \), \( b = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} \)

3. Find least squares solutions for each of the following linear systems:

(a) \[
\begin{bmatrix}
1 & 1 \\
0 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 & 3 \\
-1 & 3 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 2 & 4 \\
-2 & -3 & -7 \\
1 & 3 & 5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 & 2 & -1 \\
3 & 5 & 5 \\
-1 & -1 & -2 \\
3 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}
\]

4. Find the orthogonal projection of \( u \) onto the subspace of \( \mathbb{R}^4 \) spanned by the vectors \( v_1, v_2, \) and \( v_3 \).
(a) \( \mathbf{u} = (6, 3, 9, 6) \); \( \mathbf{v}_1 = (2, 1, 1, 1) \); \( \mathbf{v}_2 = (1, 0, 1, 1) \); \( \mathbf{v}_3 = (-2, -1, 0, -1) \)

(b) \( \mathbf{u} = (-2, 0, 2, 4) \); \( \mathbf{v}_1 = (1, 1, 3, 0) \); \( \mathbf{v}_2 = (-2, -1, -2, 1) \); \( \mathbf{v}_3 = (-3, -1, 1, 3) \)

Answer to Exercise 6.3

1. (a) \[
\begin{bmatrix}
21 & 25 \\
25 & 35
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
20 \\
20
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
15 & -1 & 5 \\
-1 & 22 & 30 \\
5 & 30 & 45
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
-1 \\
9 \\
13
\end{bmatrix}
\]

2. (a) \( x_1 = 5 \), \( x_2 = \frac{1}{7} \); \[
\begin{bmatrix}
\frac{11}{2} \\
-\frac{9}{2} \\
-4
\end{bmatrix}
\]

(b) \( x_1 = \frac{3}{7} \), \( x_2 = -\frac{2}{3} \); \[
\begin{bmatrix}
\frac{46}{21} \\
\frac{5}{21} \\
\frac{13}{21}
\end{bmatrix}
\]

(c) \( x_1 = 12 \), \( x_2 = -3 \), \( x_3 = 9 \); \[
\begin{bmatrix}
3 \\
3 \\
9
\end{bmatrix}
\]

(d) \( x_1 = 14 \), \( x_2 = 30 \), \( x_3 = 26 \); \[
\begin{bmatrix}
2 \\
6 \\
-2 \\
4
\end{bmatrix}
\]

3. (a) \( \mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \) (b) \( \mathbf{x} = \begin{bmatrix} -\frac{1}{37} \\ \frac{21}{37} \end{bmatrix} \) (c) \( \mathbf{x} = \begin{bmatrix} -\frac{85}{33} \\ \frac{130}{33} \\ -\frac{40}{33} \\ \frac{1}{33} \end{bmatrix} \) (d) \( \mathbf{x} = \begin{bmatrix} -\frac{23}{6} \\ \frac{7}{6} \\ \frac{1}{6} \end{bmatrix} \)

4. (a) \( (7, 2, 9, 5) \) (b) \( (-\frac{12}{5}, -\frac{4}{5}, \frac{12}{5}, \frac{16}{5}) \)

6.4 Complex Inner Product Space

**Definition 6.6.**

An inner product on a complex vector space \( V \) is a function that associates a complex number \( \langle \mathbf{u}, \mathbf{v} \rangle \) with each pair of vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( V \) in such a way that following conditions are satisfied:

(i) \( \langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle} \) for all \( \mathbf{u} \) and \( \mathbf{v} \) in \( V \).

(ii) \( \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \) for all \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) in \( V \).

(iii) \( \langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle \) for all \( \mathbf{u}, \mathbf{v} \) in \( V \) and all scalars \( \alpha \).

(iv) \( \langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \) with equality if and only if \( \mathbf{u} = \mathbf{0} \).

A complex vector space with an inner product is called a complex inner product space.

The following additional properties follow immediately from the four inner product conditions:

(i) \( \langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0 \)

(ii) \( \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \)
(iii) \( \langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle \)

**Example 6.23.**

If \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) are vectors in \( \mathbb{C}^n \), then we can show that the inner product

\[
\langle u, v \rangle = u \cdot v = u^T v = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}
\]

satisfies the four inner product conditions. This inner product is known as the **Euclidean inner product on** \( \mathbb{C}^n \).

**Example 6.24.**

If \( U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \) and \( V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \) are any \( 2 \times 2 \) matrices with complex entries, then the following formula defines a complex inner product on complex \( M_{2\times2} \):

\[
\langle U, V \rangle = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3} + u_4\overline{v_4}
\]

For example, if \( U = \begin{bmatrix} 0 & i \\ 1 & 1 + i \end{bmatrix} \) and \( V = \begin{bmatrix} 1 & -i \\ 0 & 2i \end{bmatrix} \), then

\[
\langle U, V \rangle = (0)(1) + i(-i) + (1)(0) + (1 + i)(-2i) = 0 + i^2 + 0 - 2i - 2i^2 = 1 - 2i
\]

**Example 6.25.**

If \( f(x) = f_1(x) + if_2(x) \) is a complex-valued function of the real variable \( x \), and if \( f_1(x) \) and \( f_2(x) \) are continuous on \([a, b]\), then we define

\[
\int_a^b f(x)dx = \int_a^b [f_1(x) + if_2(x)]dx = \int_a^b f_1(x)dx + i \int_a^b f_2(x)dx
\]

We can show that if the functions \( f = f_1(x) + if_2(x) \) and \( g = g_1(x) + ig_2(x) \) are vectors in complex \( C[a, b] \), then the following formula defines an inner product on complex \( C[a, b] \):

\[
\langle f, g \rangle = \int_a^b [f_1(x) + if_2(x)] [g_1(x) + ig_2(x)]dx
\]

\[
= \int_a^b [f_1(x) + if_2(x)][g_1(x) - ig_2(x)]dx
\]

\[
= \int_a^b [f_1(x)g_1(x) + f_2(x)g_2(x)]dx + i \int_a^b [f_2(x)g_1(x) - f_1(x)g_2(x)]dx
\]

\( ♣ \)
In complex inner product spaces, as in real inner product spaces, the norm (or length) of a vector $\mathbf{u}$ is defined by

$$\|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{1/2}$$

and the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

**Example 6.26.**

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{C}^n$ with the Euclidean inner product, then

$$\|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{1/2} = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}$$

and

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v})^{1/2}$$

$$= \sqrt{|u_1 - v_1|^2 + |u_2 - v_2|^2 + \cdots + |u_n - v_n|^2}$$

**Unitary, Normal, and Hermitian Matrices**

If $A$ is a matrix with complex entries, then the conjugate transpose of $A$, denoted by $A^*$, is defined by

$$A^* = \overline{A^T}$$

where $\overline{A}$ is the matrix whose entries are the complex conjugates of the corresponding entries in $A$ and $\overline{A}^T$ is the transpose of $\overline{A}$. The conjugate transpose is also called the Hermitian transpose.

**Example 6.27.**

If

$$A = \begin{bmatrix} 1 + i & -i \\ 2 & 3 - 2i \\ i \end{bmatrix}$$

then

$$\overline{A} = \begin{bmatrix} 1 - i & i \\ 2 & 3 + 2i \\ -i \end{bmatrix}$$

so

$$A^* = \overline{A}^T = \begin{bmatrix} 1 - i & 2 \\ -i & 3 + 2i \\ 0 & -i \end{bmatrix}$$
If $A$ and $B$ are matrices with complex entries and $k$ is any complex number, then

(a) $(A^*)^* = A$
(b) $(A + B)^* = A^* + B^*$
(c) $(kA)^* = \overline{k}A^*$
(d) $(AB)^* = B^*A^*$

A square matrix $A$ with complex entries is called \textit{unitary} if

$$A^{-1} = A^*.$$

If $A$ is an $n \times n$ matrix with complex entries, then the following are equivalent.

(a) $A$ is unitary.
(b) The row vectors of $A$ form an orthonormal set in $\mathbb{C}^n$ with the Euclidean inner product.
(c) The column vectors of $A$ form an orthonormal set in $\mathbb{C}^n$ with the Euclidean inner product.
The matrix

\[
A = \begin{bmatrix}
\frac{1+i}{2} & \frac{1+i}{2} \\
\frac{1-i}{2} & -\frac{1+i}{2}
\end{bmatrix}
\]  

(6.18)

has row vectors

\[
r_1 = \left( \frac{1+i}{2}, \frac{1+i}{2} \right) \quad \text{and} \quad r_2 = \left( \frac{1-i}{2}, -\frac{1+i}{2} \right).
\]

Relative to the Euclidean inner product on \( \mathbb{C}^2 \), we have

\[
\|r_1\| = \sqrt{\frac{(1+i)^2}{2} + \frac{(1+i)^2}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1
\]

\[
\|r_2\| = \sqrt{\frac{(1-i)^2}{2} + \frac{(-1+i)^2}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1
\]

and

\[
r_1 \cdot r_2 = \left( \frac{1+i}{2} \right) \left( \frac{1-i}{2} \right) + \left( \frac{1+i}{2} \right) \left( -\frac{1+i}{2} \right)
\]

\[
= \left( \frac{1+i}{2} \right) \left( \frac{1-i}{2} \right) + \left( \frac{1+i}{2} \right) \left( \frac{-1-i}{2} \right)
\]

\[
= \frac{i}{2} - \frac{i}{2} = 0
\]

so the row vectors form an orthonormal set in \( \mathbb{C}^2 \). Thus \( A \) is unitary and

\[
A^{-1} = A^* = \begin{bmatrix}
\frac{1-i}{2} & \frac{1+i}{2} \\
\frac{1-i}{2} & -\frac{1-i}{2}
\end{bmatrix}
\]

(6.19)

We can verify that matrix (6.19) is the inverse of matrix (6.18) by showing that \( AA^* = A^*A = I \).

Example 6.28.

A square matrix \( A \) with complex entries is called \textbf{Hermitian} if

\[
A = A^*.
\]
Example 6.29.

If

\[
A = \begin{bmatrix}
1 & i & 1 + i \\
-i & -5 & 2 - i \\
1 - i & 2 + i & 3
\end{bmatrix}
\]

then

\[
A^* = \overline{A}^T = \begin{bmatrix}
1 & -i & 1 - i \\
i & -5 & 2 + i \\
1 + i & 2 - i & 3
\end{bmatrix},
\]

so

\[
A^* = A^T = \begin{bmatrix}
1 & i & 1 + i \\
-i & -5 & 2 - i \\
1 - i & 2 + i & 3
\end{bmatrix} = A
\]

which mean that A is Hermitian. ♣

Definition 6.9.

A square matrix A with complex entries is called normal if

\[
AA^* = A^*A
\]

Example 6.30.

Every Hermitian matrix A is normal since

\[
AA^* = AA = A^*A,
\]

and every unitary matrix A is normal since

\[
AA^* = I = A^*A. \quad ♣
\]

Theorem 6.19.

The eigenvalues of a Hermitian matrix are real numbers.

Example 6.31.

Let \( A = \begin{bmatrix}
2 & 1 + i \\
1 - i & 3
\end{bmatrix} \). Find the eigenvalues A.

Solution
Exercise 6.4

1. Let \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). Show that \( \langle u, v \rangle = 3u_1 \bar{v}_1 + 2u_2 \bar{v}_2 \) defines an inner product on \( \mathbb{C}^2 \)?

2. Compute \( \langle u, v \rangle \) using the inner product in Exercise 1.
   
   (a) \( u = (2i, -i), \ v = (-i, 3i) \)
   
   (b) \( u = (0, 0), \ v = (1 - i, 7 - 5i) \)
   
   (c) \( u = (1 + i, 1 - i), \ v = (1 - i, 1 + i) \)
   
   (d) \( u = (3i, -1 + 2i), \ v = (3i, -1 + 2i) \)

3. Compute \( \langle u, v \rangle \) using the inner product defined by
   
   \( \langle u, v \rangle = u_1 \bar{v}_1 + (1 + i) u_1 \bar{v}_2 + (1 - i) u_2 \bar{v}_1 + 3u_2 \bar{v}_2. \)
   
   (a) \( u = (2i, -i), \ v = (-i, 3i) \)
   
   (b) \( u = (0, 0), \ v = (1 - i, 7 - 5i) \)
   
   (c) \( u = (1 + i, 1 - i), \ v = (1 - i, 1 + i) \)
   
   (d) \( u = (3i, -1 + 2i), \ v = (3i, -1 + 2i) \)

4. Use the inner product of Example 6.24 to find \( \langle U, V \rangle \) if
   
   \[ U = \begin{bmatrix} -i & 1 + i \\ 1 - i & i \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 3 & -2 - 3i \\ 4i & 1 \end{bmatrix} \]

5. Let \( V \) be the vector space of complex-valued functions of the real variable \( x \), and let \( f = f_1(x) + if_2(x) \) and \( g = g_1(x) + ig_2(x) \) be vectors in \( V \). Does
   
   \[ \langle f, g \rangle = \overline{[f_1(0) + if_2(0)]} \overline{[g_1(0) + ig_2(0)]} \]
   
   define an inner product on \( V \)? If not, list all conditions that fail to hold.

6. Use the Euclidean inner product to find \( \|w\| \) if
   
   (a) \( w = (-i, 3i) \)
   
   (b) \( w = (1 - i, 1 + i) \)
   
   (c) \( w = (0, 2 - i) \)
   
   (d) \( w = (0, 0) \)

7. Use the inner product of Example 6.24 to find \( \|A\| \) if
   
   (a) \( A = \begin{bmatrix} -i & 7i \\ 6i & 2i \end{bmatrix} \)
   
   (b) \( A = \begin{bmatrix} -1 & 1 + i \\ 1 - i & 3 \end{bmatrix} \)

8. Use the Euclidean inner product on \( \mathbb{C}^2 \) to find \( d(x, y) \) if
   
   (a) \( x = (1, 1), \ y = (i, -i) \)
   
   (b) \( x = (1 - i, 3 + 2i), \ y = (1 + i, 3) \)

9. Let complex \( M_{2\times 2} \) have the inner product of Example 6.24. Find \( d(A, B) \) if
(a) \( A = \begin{bmatrix} i & 5i \\ 8i & 3i \end{bmatrix} \) and \( B = \begin{bmatrix} -5i & 0 \\ 7i & -3i \end{bmatrix} \)

(b) \( A = \begin{bmatrix} -1 & 1 - i \\ 1 + i & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 2i & 2 - 3i \\ i & 1 \end{bmatrix} \)

10. In each part, find \( A^* \).

(a) \( A = \begin{bmatrix} 2i & 1 - i \\ 4 & 3 + i \\ 5 + i & 0 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 2i & 1 - i & -1 + i \\ 4 & 5 - 7i & -i \\ i & 3 & 1 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 7i & 0 & -3i \end{bmatrix} \)

(d) \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \)

11. Find \( k, \ell, \) and \( m \) to make \( A \) a Hermitian matrix if \( A = \begin{bmatrix} -1 & k & -i \\ 3 - 5i & 0 & m \\ \ell & 2 + 4i & 2 \end{bmatrix} \).

12. Use Theorem 6.18 to determine which of the following are unitary matrices.

(a) \( \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \)

(b) \( \begin{bmatrix} i \sqrt{2} & 1 \sqrt{2} \\ -i \sqrt{2} & 1 \sqrt{2} \end{bmatrix} \)

(c) \( \begin{bmatrix} 1 + i & 1 + i \\ 1 - i & -1 + i \end{bmatrix} \)

(d) \( \begin{bmatrix} -i \sqrt{2} & i \sqrt{6} & i \sqrt{3} \\ i \sqrt{2} & -i \sqrt{6} & i \sqrt{3} \\ -i \sqrt{2} & i \sqrt{6} & i \sqrt{3} \end{bmatrix} \)

13. In each part, verify that the matrix is unitary and find its inverse.

(a) \( \begin{bmatrix} 3 & 4 \sqrt{5} \\ 5 & 5 \sqrt{5} \\ -4 \sqrt{5} & 3 \sqrt{5} \end{bmatrix} \)

(b) \( \begin{bmatrix} 1 \sqrt{2} & 1 \sqrt{2} \\ 1 + i \sqrt{2} & 1 + i \sqrt{2} \end{bmatrix} \)

14. Find the eigenvalues of the following matrices.

(a) \( A = \begin{bmatrix} 4 & 1 - i \\ 1 + i & 5 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 6 & 2 + 2i \\ 2 - 2i & 4 \end{bmatrix} \)

(c) \( \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1 + i \\ 0 & -1 - i & 0 \end{bmatrix} \)

15. (a) Find a \( 2 \times 2 \) matrix that is both Hermitian and unitary and whose entries are not all real numbers.

(b) What can you say about the inverse of a matrix that is both Hermitian and unitary?
Answer to Selected Exercise 6.4

2. (a) $-12$  (b) 0  (c) $2i$  (d) 37  
3. (a) $4+5i$ (b) 0  (c) $4-4i$  (d) 42  
4. $9-5i$  
5. No. Condition 4 fails.  
6. (a) $\sqrt{10}$  (b) 2  (c) $\sqrt{5}$  (d) 0  
7. (a) $3\sqrt{10}$  (b) $\sqrt{14}$  
8. (a) 2  (b) $2\sqrt{2}$  
9. (a) $7\sqrt{2}$  (b) $2\sqrt{3}$  

10. (a) $\begin{bmatrix} -2i & 4 & 5-i \\ 1+i & 3-i & 0 \end{bmatrix}$  
    (b) $\begin{bmatrix} -2i & 4 & -i \\ 1+i & 5+7i & 3 \\ -1-i & i & 1 \end{bmatrix}$  
    (c) $\begin{bmatrix} -7i \\ 0 \\ 3i \end{bmatrix}$  
    (d) $\begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \\ \bar{a}_{13} & \bar{a}_{23} \end{bmatrix}$  

11. $k = 3 + 5i$, $l = i$, $m = 2 - 4i$  
12. (a), (b)  

13. (a) $A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$  
    (b) $A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{1-i}{2} \end{bmatrix}$  

14. (a) $\lambda = 3, 6$  (b) $\lambda = 2, 8$  (c) $\lambda = -2, 1, 5$  

15. (a) $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is one possibility.